Week 2 Review

Tuesday, January 28, 2020 3:48 PM

Spring 2020 Math 152

Week 2 in Review

courtesy: David J. Manuel

(covering 5.5, 6.1, and 6.2)

(Problems with a * beside them will also be done in Python)

1 Section 5.5

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

Let $u = g(x)$ (stuff in parentheses)

$du = g'(x)\,dx$

$dx = \frac{du}{g'(x)}$

$$= \int f'(u)\, \cancel{g'(x)}\, \frac{du}{\cancel{g'(x)}}$$

1. Evaluate the following integrals:

(a) $\displaystyle\int_0^2 \frac{dx}{(3x+2)^2}$

Let $u = 3x+2$

$du = 3\,dx$

$dx = \frac{du}{3}$

if $x=2$, $u = 3(2)+2 = 8$

if $x=0$, $u = 3(0)+2 = 2$

Leave boundaries as $x=0$ and $x=2$ (SPECIFY!) or change as done here

$$= \int_2^8 \frac{du/3}{u^2}$$

$$= \frac{1}{3}\int_2^8 u^{-2}\,du$$

$$= -\frac{1}{3}u^{-1}\Big|_2^8$$

$$= -\frac{1}{3}\left(\frac{1}{8}\right) + \frac{1}{3}\left(\frac{1}{2}\right) = \boxed{\frac{1}{8}}$$

(b) $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x \cdot du$

$$= \int \frac{\cos(u)}{\cancel{x}} \cdot \cancel{x}\, du$$

$$= \int \cos(u)\,du$$

$= \sin(u) + C$ (indefinite integral, so have to change back to x)

$$= \sin(\ln x) + C$$

(c) $\int_0^1 x e^{(-x^2)} dx^*$

Let $u = -x^2$

$du = -2x\, dx$

$dx = \frac{du}{-2x}$

if $x=1$, $u = -(1)^2 = -1$

if $x=0$, $u = -(0)^2 = 0$

NOTE exponents **FIRST**

$= \int_0^{-1} \cancel{x} e^u \frac{du}{-2\cancel{x}}$

$= -\frac{1}{2}\int_0^{-1} e^u\, du$ can change to $+\frac{1}{2}\int_{-1}^{0} e^u\, du$ if you want

$= -\frac{1}{2} e^u \Big|_0^{-1}$

$= -\frac{1}{2}\left(e^{-1} - e^0\right) = \boxed{-\frac{1}{2}\left(e^{-1} - 1\right)}$

In [2]:
```
from sympy import *
```

In [3]:
```
x=symbols('x')
f=x*exp(-x**2)
integrate(f,(x,0,1))
```

Out[3]: $\frac{1}{2} - \frac{1}{2e}$

$e^{2x} = (e^x)^2$

(d) $\int_0^{\frac{1}{2}\ln 3} \frac{e^x}{e^{2x}+1} dx$

Let $u = e^x$

$du = e^x\, dx$

$dx = \frac{du}{e^x}$

(NOTE if $u = e^{2x}+1$, $du = 2e^{2x}$ does not cancel correctly!)

if $x = \frac{1}{2}\ln 3$, $u = e^{\frac{1}{2}\ln 3} = \cancel{e}^{\cancel{\ln} 3^{1/2}} = \sqrt{3}$

if $x = 0$, $u = e^0 = 1$

$= \int_1^{\sqrt{3}} \frac{\cancel{e^x}}{u^2+1} \frac{du}{\cancel{e^x}}$

$= \int_1^{\sqrt{3}} \frac{1}{u^2+1}\, du$

$= \tan^{-1}(u) \Big|_1^{\sqrt{3}}$

$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$

$= \frac{\pi}{3} - \frac{\pi}{4} = \boxed{\frac{\pi}{12}}$

$\frac{\sqrt{3}/2}{1/2}$ sin / cos

(e) $\int x^3 \sqrt{x^2+1}\, dx$

Let $u = x^2+1 \longrightarrow x^2 = u-1$

$du = 2x\, dx$

$dx = \frac{du}{2x}$

$= \int x^{\cancel{3}2} \sqrt{u} \frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int x^2 \sqrt{u}\, du$

$= \frac{1}{2}\int (u-1)\sqrt{u}\, du$ multiply out

$= \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right) du$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$

$= \boxed{\frac{1}{2}\left(\frac{2}{5}(x^2+1)^{5/2} - \frac{2}{3}(x^2+1)^{3/2}\right) + C}$

2 Section 6.1

Area Between Curves:

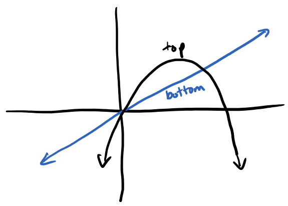
1) Graph the region
2) Find the points of intersection
3) Integrate to find area:

$\int (\text{Top} - \text{Bottom})\,dx$

$\int (\text{Right} - \text{Left})\,dy$

down parabola   line

1. Find the area bounded by the graph of $y = 6x - x^2$ and the line $y = 2x$



Intersections: $6x - x^2 = 2x$

$0 = x^2 - 4x$

$0 = x(x-4)$

$x = 0,\ x = 4$

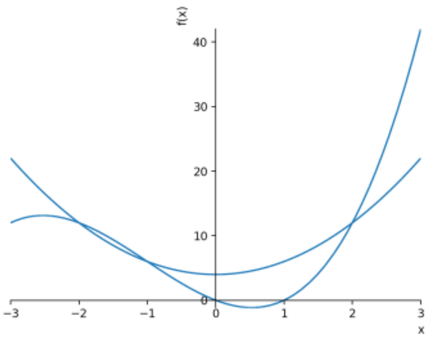
$A = \int_0^4 \left((6x - x^2) - (2x)\right) dx$

$= \int_0^4 (4x - x^2)\,dx$

$= 2x^2 - \frac{1}{3}x^3 \Big|_0^4$

$= \left(2(4)^2 - \frac{1}{3}(4^3)\right) - (0) = \boxed{\frac{32}{3}}$

2. Find the area bounded by the graphs of $y = x^3 + 3x^2 - 4x$ and $y = 2x^2 + 4$.



Intersections: $x^3 + 3x^2 - 4x = 2x^2 + 4$

$x^3 + x^2 - 4x - 4 = 0$ Factor by Grouping

$x^2(x+1) - 4(x+1) = 0$

$(x^2 - 4)(x+1) = 0$

$(x+2)(x-2)(x+1) = 0$

$x = -2,\ x = 2,\ x = -1$ Boundaries change, but cannot solve for x, so still do $y = f(x)$ (Two integrals)

$$A = \int_{-2}^{-1} \left((x^3+3x^2-4x) - (2x^2+4)\right) dx + \int_{-1}^{2} \left((2x^2+4) - (x^3+3x^2-4x)\right) dx$$

$$= \int_{-2}^{-1} (x^3 + x^2 - 4x - 4)\, dx + \int_{-1}^{2} (-x^3 - x^2 + 4x + 4)\, dx$$

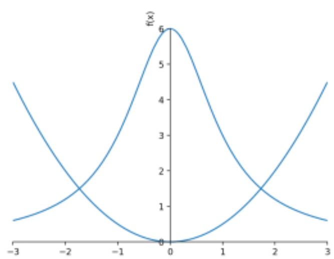
$$= \left(\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - 2x^2 - 4x \right|_{-2}^{-1} + \left(-\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 + 2x^2 + 4x\right|_{-1}^{2}$$

$$= \left(\tfrac{1}{4} + \tfrac{1}{3} - 2 + 4\right) - \left(4 - \tfrac{8}{3} - 8 + 8\right) + \left(-4 - \tfrac{8}{3} + 8 + 8\right) - \left(-\tfrac{1}{4} + \tfrac{1}{3} + 2 - 4\right)$$

$$= \tfrac{7}{12} + \tfrac{45}{4} = \boxed{\tfrac{71}{6}}$$

3. Find the area bounded by the curves $y = \dfrac{6}{1+x^2}$ and $y = \dfrac{1}{2}x^2$. *

(Graph would be given here):



Intersect: $\dfrac{6}{1+x^2} \bcancel{=} \dfrac{x^2}{2}$

$12 = x^2 + x^4$ $u = x^2$, $u^2 = x^4$, $u^2 + u - 12$

$0 = x^4 + x^2 - 12$

$0 = (x^2+4)(x^2-3)$

$x = \pm\sqrt{3}$

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{6}{1+x^2} - \frac{1}{2}x^2\right) dx$$ symmetric, can do this...

$$= 2\int_{0}^{\sqrt{3}} \left(\frac{6}{1+x^2} - \frac{1}{2}x^2\right) dx$$

$$= 2\left(6\tan^{-1} x - \tfrac{1}{6}x^3 \right|_{0}^{\sqrt{3}}$$

$$= 12\tan^{-1}(\sqrt{3}) - \tfrac{1}{3}(\sqrt{3})^3$$

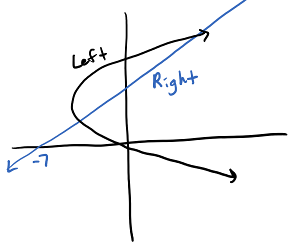
$$= 12\left(\tfrac{\pi}{3}\right) - \tfrac{3\sqrt{3}}{3} = \boxed{4\pi - \sqrt{3}}$$

```
x=symbols('x')
f=6/(1+x**2)
g=Rational(1,2)*x**2 #Keep fractions exact
# plot((f,(x,-3,3)),(g,(x,-3,3)))  used to create graph at the left.  Saving space here.
intersects=solve(f-g,x)
print(intersects)
Area=integrate(f-g,(x,-sqrt(3),sqrt(3)))
print('The area is',Area)
```

```
[-sqrt(3), sqrt(3), -2*I, 2*I]
The area is -sqrt(3) + 4*pi
```

(Changed) easier to solve for $x = g(y)$

4. Find the area of the region bounded by $x - 2y = {}^{-}7$ and $y^2 - 6y - x = 0$.



$x = {}^{-}7 + 2y$ line x-int $(-7,0)$ positive slope

$x = y^2 - 6y$ parabola opens right

Intersections: $-7 + 2y = y^2 - 6y$

$0 = y^2 - 8y + 7$

$0 = (y-7)(y-1)$

$y = 7,\ y = 1$

$$A = \int_1^7 \left((-7+2y) - (y^2 - 6y)\right) dy$$

$$= \int_1^7 (-y^2 + 8y - 7)\, dy$$

$$= -\tfrac{1}{3}y^3 + 4y^2 - 7y \Big|_1^7$$

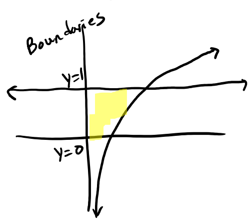
$$= \left(-\tfrac{1}{3}\cdot 7^3 + 4(7)^2 - 7(7)\right) - \left(-\tfrac{1}{3} + 4 - 7\right)$$

$$= \frac{-343}{3} + 147 + \frac{1}{3} + 3 = \boxed{36}$$

DO NOT KNOW HOW TO INTEGRATE $y = \ln x$ SO re write as $x = e^y$

5. Find the area in the first quadrant to the left of $y = \ln x$ and below $y = 1$.

$x = e^y$



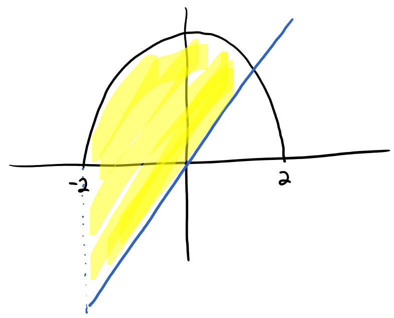
$$A = \int_0^1 e^y\, dy$$

$$= e^y \Big|_0^1$$

$$= e^1 - e^0 = \boxed{e - 1}$$

6. Sketch a region whose area is represented by the integral $\int_{-2}^{\sqrt{2}} (\sqrt{4 - x^2} - x)\, dx$.

TOP semicircle BOTTOM line



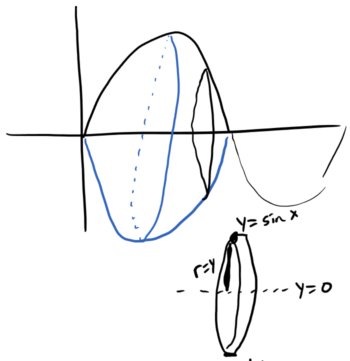
3 Section 6.2

1) Graph region you are rotating (reflect to "see" solid)
2) Find intersection points
3) Volume Using Cylindrical Shells

$$V = \int_a^b \left(\pi\,(\text{Outer})^2 - \pi\,(\text{Inner})^2\right) \begin{matrix} dx \\ \text{or} \\ dy \end{matrix}$$

$dx \to$ rotating about x-axis
$dy \to$ rotating about y-axis

1. Find the volume of the solid formed by rotating the region above the x-axis (closest to the origin) bounded by the curves $y = \sin x$ and $y = 0$ about the x-axis.



$V = \pi r^2 h$
$= \pi y^2\, dx$

No hole in solid ⇒ No "inner" radius

Intersections: $\sin x = 0$

$x = 0, \pi$

Identity: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Based on $\cos^2 x + \sin^2 x = 1$
$-(\cos^2 x - \sin^2 x = \cos(2x))$

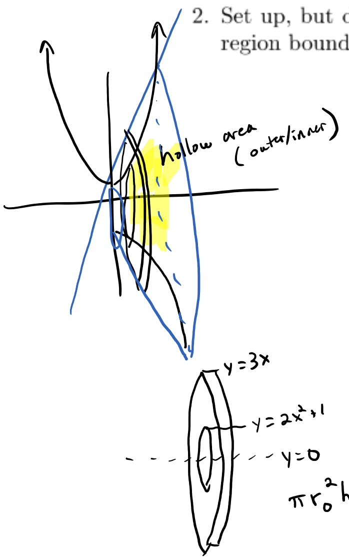
$$V = \int_0^{\pi} \pi (\sin x)^2\, dx$$
$$= \pi \int_0^{\pi} \sin^2 x\, dx$$
$$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos(2x))\, dx$$

$u = 2x$
$du = 2\,dx$
$dx = \frac{1}{2} du$

$$= \frac{\pi}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi}$$
$$= \frac{\pi}{2}\left(\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(2\cdot 0)\right)\right)$$
$$= \boxed{\frac{\pi^2}{2}}$$

2. Set up, but do not evaluate, an integral to find the volume of the solid formed by rotating the region bounded by $y = 2x^2 + 1$ and $y = 3x$ about the x-axis.

parabola opens up   line



$\pi r_o^2 h - \pi r_i^2 h$

Intersections: $2x^2 + 1 = 3x$

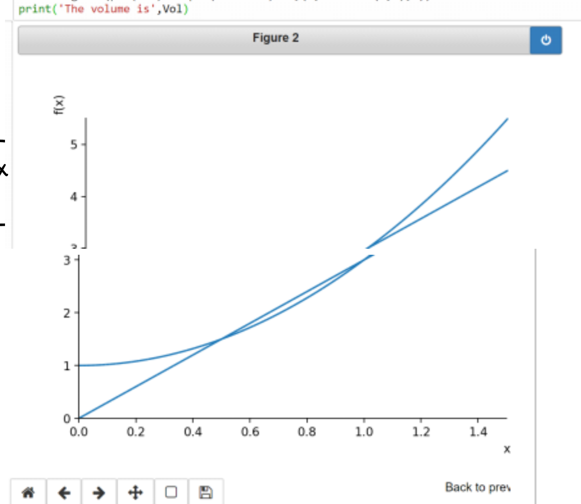
$2x^2 - 3x + 1 = 0$

$(2x-1)(x-1) = 0$

$x = \frac{1}{2} \quad x = 1$

$$\boxed{V = \int_{\frac{1}{2}}^{1} \left(\pi(3x)^2 - \pi(2x^2+1)^2\right) dx}$$

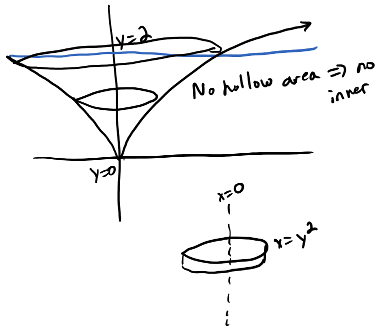
```
x=symbols('x')
f=3*x
g=2*x**2+1
plot((f,(x,0,1.5)),(g,(x,0,1.5)))
intersects=solve(f-g,x)
print(intersects)
Vol=integrate(pi*(3*x)**2-pi*(2*x**2+1)**2,(x,Rational(1,2),1))
print('The volume is',Vol)
```




```
[1/2, 1]
The volume is 11*pi/60
```

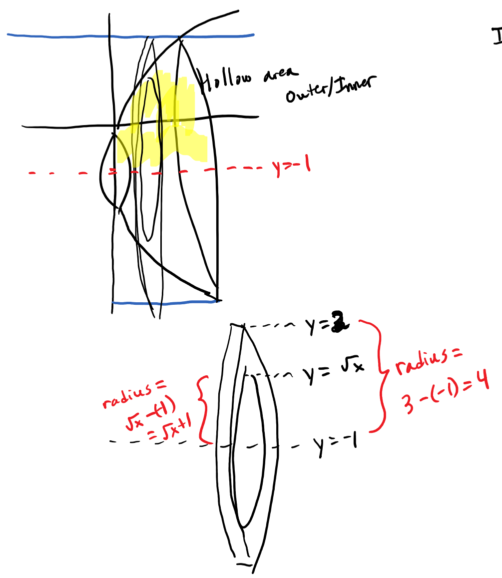
$x = y^2$

3. Find the volume of the solid formed by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 0$, and $y = 2$ about the y-axis. dy



$$V = \int_0^2 \pi (y^2)^2 \, dy$$
$$= \int_0^2 \pi y^4 \, dy$$
$$= \frac{\pi}{5} y^5 \Big|_0^2$$
$$= \boxed{\frac{32\pi}{5}}$$

4. Find the volume of the solid formed by rotating the region in the previous example about the line $y = -1$. Horizontal ("x" axis) ⇒ dx



Intersection: $\sqrt{x} = 2$

$x = 4$

$$V = \int_0^4 \left(\pi\left(2-(-1)\right)^2 - \pi\left(\sqrt{x}-(-1)\right)^2\right) dx$$
$$= \int_0^4 9\pi - \pi\left(x + 2\sqrt{x} + 1\right)) \, dx$$
$$= \int_0^4 8\pi - \pi x - 2\pi x^{1/2}) \, dx$$
$$= 8\pi x - \frac{\pi}{2} x^2 - 2\pi\left(\frac{2}{3}\right) x^{3/2} \Big|_0^4$$
$$= \left(32\pi - 8\pi - \frac{4\pi}{3} \cdot 4^{3/2}\right) - 0$$
$$= \boxed{\frac{40\pi}{3}}$$

5. Find the volume of a square pyramid whose height is h and whose base is s by s.

Next Week

6. The base of a solid is the unit circle in the x-y plane. Cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume of the solid.

Next week