Week 5 Review

Tuesday, February 25, 2020 6:19 PM

1 Section 7.3

1. Evaluate the following integrals:

$x=2 \quad \theta=\sin^{-1}\left(\frac{2}{2}\right)=\frac{\pi}{2}$

$x=-2 \quad \theta=\sin^{-1}\left(\frac{-2}{2}\right)=\frac{-\pi}{2}$

(a) $\int_{-2}^{2}\sqrt{4-x^2}\,dx$

$a^2-x^2 \Rightarrow x=a\sin\theta$

$x=2\sin\theta$

$dx=2\cos\theta\,d\theta$ ☆☆

$=\int_{-\frac{\pi}{2}}^{\pi/2}\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$

$=\int_{-\frac{\pi}{2}}^{\pi/2}\sqrt{4(\underbrace{1-\sin^2\theta}_{\cos^2\theta})}\,2\cos\theta\,d\theta$

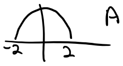
$=4\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$ $\qquad \cos^2\theta=\frac{1}{2}(1+\cos(2\theta))$

$=2\int_{-\frac{\pi}{2}}^{\pi/2}1+\cos(2\theta)\,d\theta$

$=2\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Big|_{-\frac{\pi}{2}}^{\pi/2}$

$=2\left(\frac{\pi}{2}+\frac{1}{2}\overset{0}{\sin\pi}-\frac{-\pi}{2}+\frac{1}{2}\overset{0}{\sin(-\pi)}\right)=\boxed{2\pi}$

NOTE: $y=\sqrt{4-x^2}$ is a semicircle



$A=\frac{1}{2}\pi r^2$

$=\frac{1}{2}\pi(2)^2=2\pi$!

(b) $\int\frac{1}{\sqrt{x^2+25}}\,dx$

$x^2+a^2 \Rightarrow x=a\tan\theta$

$x=5\tan\theta$

$dx=5\sec^2\theta\,d\theta$

Return to x

$=\int\frac{1}{\sqrt{25\tan^2\theta+25}}\cdot 5\sec^2\theta\,d\theta$

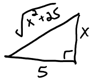
$=\int\frac{1}{\sqrt{\cancel{25}(\underbrace{\tan^2\theta+1}_{\sec^2\theta})}}\cdot\cancel{5}\sec^2\theta\,d\theta$

$=\int\sec\theta\,d\theta$

$=\ln|\sec\theta+\tan\theta|+C$

$=\boxed{\ln\left|\frac{\sqrt{x^2+25}}{5}+\frac{x}{5}\right|+C}$

$\tan\theta=\frac{x}{5}\begin{matrix}\text{Opp}\\\text{Adj}\end{matrix}$



$\sec\theta=\frac{1}{\cos\theta}=\frac{\sqrt{x^2+25}}{5}$

(c) $\int\frac{x^3}{\sqrt{4x^2-9}}\,dx=\int\frac{x^3}{2\sqrt{x^2-\frac{9}{4}}}\,dx$

$x^2-a^2\Rightarrow x=a\sec\theta$

$x=\frac{3}{2}\sec\theta$

$dx=\frac{3}{2}\sec\theta\tan\theta$

$=\int\frac{\frac{27}{8}\sec^3\theta}{2\sqrt{\frac{9}{4}\sec^2\theta-\frac{9}{4}}}\cdot\frac{3}{2}\sec\theta\tan\theta$

$=\frac{81}{32}\int\frac{\sec^4\theta\tan\theta}{\sqrt{\frac{9}{4}(\underbrace{\sec^2\theta-1}_{\tan^2\theta})}}$

$=\frac{27}{16}\int\sec^4\theta\,d\theta$

$=\frac{27}{16}\int\sec^2\theta(\tan^2\theta+1)\,d\theta$ $\qquad u=\tan\theta$, $du=\sec^2\theta\,d\theta$

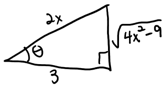
$=\frac{27}{16}\int(u^2-1)\,du$

$=\frac{27}{16}\left(\frac{1}{3}u^3-u\right)+C$

$=\frac{27}{16}\left(\frac{1}{3}\tan^3\theta-\tan\theta\right)+C$

$=\boxed{\frac{27}{16}\left(\frac{1}{3}\cdot\frac{(4x^2-9)^{3/2}}{27}-\frac{\sqrt{4x^2-9}}{3}\right)+C}$

$\sec\theta=\frac{2x}{3}\begin{matrix}\text{Hyp}\\\text{Adj}\end{matrix}$



(d) $\int\frac{x^2}{\sqrt{9-x^2}}\,dx^*$

$x=3\sin\theta$

$dx=3\cos\theta\,d\theta$

$=\int\frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\cdot 3\cos\theta\,d\theta$

$=27\int\frac{\sin^2\theta\cos\theta}{\sqrt{9(\underbrace{1-\sin^2\theta}_{\cos^2\theta})}}\,d\theta$

$=9\int\sin^2\theta\,d\theta$ $\qquad \sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$

$=\frac{9}{2}\int(1-\cos(2\theta))\,d\theta$

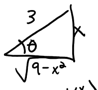
$=\frac{9}{2}\left(\theta-\frac{1}{2}\sin(\underline{2\theta})\right)+C$

Use double angle identity

$=\frac{9}{2}\left(\theta-\frac{1}{2}\cdot 2\sin\theta\cos\theta\right)+C$

$=\boxed{\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right)-\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right)+C}$

$\sin\theta=\frac{x}{3}\begin{matrix}\text{Opp}\\\text{Hyp}\end{matrix}$



$\theta=\sin^{-1}\left(\frac{x}{3}\right)$

$x=1\rightarrow\theta=\tan^{-1}(1)=\pi/4$

$x=0\rightarrow\theta=\tan^{-1}(0)=0$

(e) $\int_0^1 x^3\sqrt{x^2+1}\,dx$

$x=\tan\theta$

$dx=\sec^2\theta\,d\theta$

$=\int_0^{\pi/4}\tan^3\theta\sqrt{\underbrace{\tan^2\theta+1}_{\sec^2\theta}}\cdot\sec^2\theta\,d\theta$

$=\int_0^{\pi/4}\tan^3\theta\cdot\sec^3\theta\,d\theta$

$=\int_0^{\pi/4}\sec\theta\tan\theta\left(\overset{\sec^2\theta-1}{\tan^2\theta}\cdot\sec^2\theta\right)d\theta$

$\theta=\frac{\pi}{4}\rightarrow u=\sec\left(\frac{\pi}{4}\right)=\sqrt{2}$

$\theta=0\rightarrow u=\sec(0)=1$

$u=\sec\theta$

$du=\sec\theta\tan\theta\,d\theta$

$=\int_1^{\sqrt{2}}(u^2-1)u^2\,du$

$=\int_1^{\sqrt{2}}(u^4-u^2)\,du$

$=\frac{1}{5}u^5-\frac{1}{3}u^3\Big|_1^{\sqrt{2}}$

$=\boxed{\left(\frac{1}{5}(\sqrt{2})^5-\frac{1}{3}(\sqrt{2})^3\right)-\left(\frac{1}{5}-\frac{1}{3}\right)}$

$\left(\frac{-2}{2}\right)^2=1$

(f) $\int\frac{1}{x^2-2x+5}\,dx$

Complete the square $x^2-2x+1+5-1$

$=(x-1)^2+4$

$=\int\frac{1}{(x-1)^2+4}\,dx$ $\qquad x-1=2\tan\theta$ (can also let $u=x-1$, then $u=2\tan\theta$)

$dx=2\sec^2\theta\,d\theta$

$=\int\frac{1}{4\tan^2\theta+4}\cdot 2\sec^2\theta\,d\theta$

$=\int\frac{1}{4(\underbrace{\tan^2\theta+1}_{\cancel{\sec^2\theta}})}\cdot 2\cancel{\sec^2\theta}\,d\theta$

$=\frac{1}{2}\int d\theta$

$=\frac{1}{2}\theta+C$

$=\boxed{\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C}$

Useful formula:

$\int\frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(Distinct Linear Factors)

(b) $\int\frac{x-3}{x^3-6x^2+5x}\,dx$

Factor !!! Correctly ☆ $\left(\frac{x-3}{x(x-5)(x-1)}=\frac{A}{x}+\frac{B}{x-5}+\frac{C}{x-1}\right)x(x-5)(x-1)$

2 Section 7.4

Linear Factors)

2 Section 7.4

1. Evaluate the following integrals:

(a) $\int \frac{x\,dx}{(x+2)(x-2)}$

(Distinct Linear Factors)

$$\left(\frac{x}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}\right)(x+2)(x-2)$$

$$x = A(x-2) + B(x+2)$$

① Expand and Match Powers:

$$x = Ax - 2A + Bx + 2B$$

$$\left.\begin{array}{l} A+B=1 \\ -2A+2B=0 \end{array}\right\} \begin{array}{l} A=\frac{1}{2} \\ B=\frac{1}{2} \end{array}$$

② Choose "Nice" Values of x

$\underline{x=2}$ $\quad 2 = A(0) + B(4)$

$B = \frac{1}{2}$

$\underline{x=-2}$ $\quad -2 = A(-4) + B(0)$

$A = \frac{1}{2}$

$$= \int\left(\frac{\frac{1}{2}}{x+2} + \frac{\frac{1}{2}}{x-2}\right)dx$$

$$= \boxed{\frac{1}{2}\ln|x+2| + \frac{1}{2}\ln|x-2| + C}$$

(b) $\int \frac{x-3}{x^3-6x^2+5x}\,dx$

(Distinct Linear Factors)

Factor Correctly!!! ☆

$$\left(\frac{x-3}{x(x-5)(x-1)} = \frac{A}{x} + \frac{B}{x-5} + \frac{C}{x-1}\right) x(x-5)(x-1)$$

$$x-3 = A(x-5)(x-1) + Bx(x-1) + Cx(x-5)$$

(Nice values work better here...)

$x=0: \quad -3 = 5A \quad A = \frac{-3}{5}$

$x=5: \quad 2 = 20B \quad B = \frac{1}{10}$

$x=1: \quad -2 = -4C \quad C = \frac{1}{2}$

$$= \int\left(\frac{-3/5}{x} + \frac{1/10}{x-5} + \frac{\frac{1}{2}}{x-1}\right)dx$$

$$= \boxed{\frac{-3}{5}\ln|x| + \frac{1}{10}\ln|x-5| + \frac{1}{2}\ln|x-1| + C}$$

$$3\overline{)\,29\,}^{\;9\,r\,2} \quad \begin{array}{r} 29 \\ \underline{27} \\ 2 \end{array}$$

$$\frac{29}{3} = 9\frac{2}{3}$$

(c) $\int \frac{(6x+7)}{x^2+4x+4}\,dx$

(Repeated Linear Factor)

$$\left(\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}\right)(x+2)^2$$

$$6x+7 = A(x+2) + B$$

(Match Powers looks easier here)

$$6x+7 = Ax + 2A + B$$

$A = 6$

$2A + B = 7 \rightarrow B = -5$

$$= \int\left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right)dx$$

$$= \boxed{6\ln|x+2| + 5(x+2)^{-1} + C}$$

***

NOTE answer is NOT a logarithm!

(d) $\int \frac{x^3}{x^2+1}\,dx$

IMPROPER fraction: do long division

$$x^2+1\overline{)\,x^3+0x^2+0x+0\,}^{\;x}$$

$$\underline{-(x^3 \qquad + x)}$$

$$-x$$

$$\frac{x^3}{x^2+1} = x + \frac{-x}{x^2+1}$$

$$= \int\left(x - \frac{x}{x^2+1}\right)dx$$

$u = x^2+1$

$du = 2x\,dx$

$$\frac{x}{x^2+1}dx = \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\cdot\frac{1}{u}du$$

$$= \boxed{\frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C}$$

(e) $\int \frac{x^4+x-24}{x^3+4x}\,dx$*

improper: divide

$$x^3+4x\overline{)\,x^4+0x^3+0x^2+x-24\,}^{\;x}$$

$$\underline{-(x^4 \qquad +4x^2)}$$

$$-4x^2+x-24$$

$$= \int\left(x + \frac{-4x^2+x-24}{x^3+4x}\right)dx$$

Partial Fractions

split -NUMERATOR ONLY!!!

$$= \int\left(x + \frac{-6}{x} + \frac{2x+1}{x^2+4}\right)dx$$

$$= \int\left(x - \frac{6}{x} + \frac{2x \overset{=du}{}}{\underset{u=x^2+4}{x^2+4}} + \frac{1}{x^2+4}\right)dx$$

→ See 7.3 #1f above

$$= \boxed{\frac{1}{2}x^2 - 6\ln|x| + \ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

(irreducible quadratic)

$$\left(\frac{-4x^2+x-24}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}\right) x(x^2+4)$$

$$-4x^2+x-24 = A(x^2+4) + (Bx+C)x$$

$$-4x^2+x-24 = Ax^2+4A+Bx^2+Cx$$

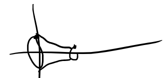
$-4 = A + B \rightarrow B = 2$

$C = 1$

$4A = -24 \rightarrow A = -6$

2. Find the volume of the solid formed by rotating the region under the curve $y = \frac{2}{x^2+3x+2}$, $x \in [0, 1]$:

(a) about the x-axis



f(x) about x-axis = SLICES

$$V = (\pi R^2 - \underset{\substack{\uparrow \\ 0}}{\pi r^2})\,dx$$

$$V = \int_0^1\left(\frac{2}{(x+2)(x+1)}\right)^2 dx$$

$$= \int_0^1 \frac{4}{(x+2)^2(x+1)^2}\,dx$$

$$\left(\frac{4}{(x+2)^2(x+1)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}\right)(x+2)^2(x+1)^2$$

$$4 = A(x+2)(x+1)^2 + B(x+1)^2 + C(x+1)(x+2)^2 + D(x+2)^2$$

$\underline{x=-2}$ $\quad 4 = B \quad$ all other terms 0

$\underline{x=-1}$ $\quad 4 = D \quad$ all other terms 0

(Choose other values - easier than multiplying out)

$x=0 \quad 4 = 2A + \underset{4}{B} + 4C + \underset{4}{4D} \rightarrow 2A + 4C = -16$

$x=1 \quad 4 = 12A + 4B + 18C + 9D \rightarrow 12A + 18C = -48$

$$\left.\begin{array}{l} A + 2C = -8 \\ 2A + 3C = -8 \end{array}\right\} \begin{array}{l} C = -8 \\ A = 8 \end{array}$$

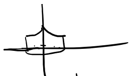
(choose method)

$$= \int_0^1\left(\frac{8}{x+2} + \frac{4}{(x+2)^2} - \frac{8}{x+1} + \frac{4}{(x+1)^2}\right)dx$$

$$= 8\ln|x+2| - \frac{4}{x+2} - 8\ln|x+1| - \frac{4}{x+1}\Big|_0^1$$

$$= \boxed{\left(8\ln(3) - \frac{4}{3} - 8\ln(2) - 2\right) - \left(8\ln(2) - 2 - 0 - 4\right)}$$

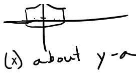
(b) about the y-axis



f(x) about y-axis = SHELLS $\quad V = 2\pi r(T - B)\,dx$

$$V = \int^1 2\pi x \cdot \frac{1}{(x+1)(x+2)}\,dx$$

(b) about the y-axis



f(x) about y-axis = SHELLS $V = 2\pi r(T - B)\,dx$

$$V = \int_0^1 2\pi x \cdot \frac{1}{(x+1)(x+2)}\,dx$$

$$= 2\pi \int_0^1 \frac{x}{(x+1)(x+2)}\,dx$$

$$\left(\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}\right)(x+1)(x+2)$$

$$x = A(x+2) + B(x+1)$$

$\underline{x = -1}$ $-1 = A$

$\underline{x = -2}$ $-2 = -B \to B = 2$

$$= 2\pi \int_0^1 \left(\frac{-1}{x+1} + \frac{2}{x+2}\right)dx$$

$$= 2\pi \left(2\ln|x+2| - \ln|x+1|\right)\Big|_0^1$$

$$= \boxed{2\pi\left(\left(2\ln(3) - \ln(2)\right) - \left(2\ln(2) - \overset{0}{\ln(1)}\right)\right)}$$