Week 7 Review

Tuesday, March 3, 2020 4:27 PM

TEXAS A&M UNIVERSITY
Math Learning Center

1 Section 7.8 Improper Integrals

Type 1 (x unbounded)

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty}\int_a^M f(x)\,dx$$

1) integrate
2) take limit

Type 2 (f(x) unbounded)

Ex $\int_{-8}^{1} x^{-1/3}\,dx$

CANNOT use Fundamental Theorem since $x^{-1/3}$ not bounded at $x=0$

$$= \int_{-8}^{0} x^{-1/3}\,dx + \int_0^1 x^{-1/3}\,dx$$

$$= \lim_{b\to 0^-}\int_{-8}^{b} x^{-1/3}\,dx + \lim_{a\to 0^+}\int_a^1 x^{-1/3}\,dx$$

$$= \lim_{b\to 0^-} \frac{3}{2}x^{2/3}\Big|_{-8}^{b} + \lim_{a\to 0^+}\frac{3}{2}x^{2/3}\Big|_a^1$$

$$= \lim_{b\to 0^-}\left(\frac{3}{2}b^{2/3} - \frac{3}{2}(-8)^{2/3}\right) + \lim_{a\to 0^+}\left(\frac{3}{2} - \frac{3}{2}a^{2/3}\right)$$

0 0

$$= -6 + \frac{3}{2} = \boxed{\frac{-9}{2}}$$

1. Evaluate the following integrals:

(a) $\int_0^\infty e^{-3x}\,dx$

$$= \lim_{M\to\infty}\int_0^M e^{-3x}\,dx$$

$$= \lim_{M\to\infty} -\frac{1}{3}e^{-3x}\Big|_0^M$$

$$= \lim_{M\to\infty} -\frac{1}{3}e^{-3M} + \frac{1}{3} = \frac{1}{3}$$

0

integral converges to $\frac{1}{3}$

(b) $\displaystyle\int_0^\infty xe^{-3x}\,dx$

Integrate by Parts

$u = x \qquad dv = e^{-3x}\,dx$

$du = dx \qquad v = -\frac{1}{3}e^{-3x}$

$$= \lim_{M\to\infty} \int_0^M x e^{-3x}\,dx$$

$$= \lim_{M\to\infty} x\left(-\frac{1}{3}e^{-3x}\right)\Big|_0^M + \frac{1}{3}\int_0^M e^{-3x}\,dx$$

$$= \lim_{M\to\infty} -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\Big|_0^M$$

$$= \lim_{M\to\infty} \underbrace{-\frac{1}{3}Me^{-3M}}_{?*} - \frac{1}{9}\underbrace{\cancel{e^{-3M}}}_{0} + \frac{1}{3}(0) + \frac{1}{9}$$

$$= \boxed{\frac{1}{9}}$$

* L'Hospital's Rule

$$-\frac{1}{3}\lim_{M\to\infty}\frac{M}{e^{3M}} = -\frac{1}{3}\lim_{M\to\infty}\frac{1}{3e^{3M}} = 0$$

(c) $\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx^*$

Integrate by Parts

$u = \ln x \qquad dv = x^{-2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -x^{-1}$

$$= \lim_{M\to\infty}\int_2^M \frac{\ln x}{x^2}\,dx$$

$$= \lim_{M\to\infty} (\ln x)\left(-\frac{1}{x}\right)\Big|_2^M + \int_2^M \frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= \lim_{M\to\infty} -\frac{\ln x}{x}\Big|_2^M + \int_2^M \frac{1}{x^2}\,dx$$

$$= \lim_{M\to\infty} -\frac{\ln x}{x} - \frac{1}{x}\Big|_2^M$$

$$= \lim_{M\to\infty} \underbrace{\frac{-\ln M}{M}}_{?*} - \underbrace{\cancel{\frac{1}{M}}}_{0} + \frac{\ln 2}{2} + \frac{1}{2}$$

$$= \boxed{\frac{\ln 2}{2} + \frac{1}{2}}$$

* L'Hospital's Rule

$$\lim_{M\to\infty} -\frac{\ln M}{M}$$

$$= \lim_{M\to\infty} -\frac{\frac{1}{M}}{1} = 0$$

In [1]:
```python
from sympy import *
```

In [2]:
```python
x=symbols('x')
f=log(x)/x**2
integrate(f,(x,2,oo))  # Use oo for infinity
```

Out[2]: $\dfrac{\log(2)}{2} + \dfrac{1}{2}$

(d) $\int_1^\infty \frac{1}{x(x^2+1)}\,dx$

Partial Fractions

$$\left(\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}\right) x(x^2+1)$$

$$1 = A(x^2+1) + (Bx+C)x$$

Multiply and match powers

$$0x^2 + 0x + 1 = Ax^2 + A + Bx^2 + Cx$$

$A+B=0$

$C=0$

$A=1 \to B=-1$

$$= \lim_{M\to\infty} \int_1^M \frac{1}{x(x^2+1)}\,dx$$

$u = x^2+1$
$du = 2x\,dx$

$$= \lim_{M\to\infty} \int_1^M \left(\frac{1}{x} - \frac{1x}{x^2+1}\right) dx$$

$$= \lim_{M\to\infty} \ln(x) - \tfrac{1}{2}\ln(x^2+1)\Big|_1^M$$

$$= \lim_{M\to\infty} \underbrace{\ln(M) - \tfrac{1}{2}\ln(M^2+1)}_{?\ \infty-\infty} - \overbrace{\ln(1)}^{=0} + \tfrac{1}{2}\ln(2)$$

? $\infty - \infty$ combine using Log Properties

$$= \lim_{M\to\infty} \overbrace{\ln\left(\frac{M}{(M^2+1)^{1/2}}\right)}^{=0} + \tfrac{1}{2}\ln(2)$$

$\frac{M}{(M^2)^{1/2}} = \frac{M}{M} = 1$

$$= \tfrac{1}{2}\ln(2)$$

(e) $\int_3^\infty \frac{x+1}{x^2-4}\,dx$

Can do Partial Fractions, etc BUT looking at dominating terms, $\frac{x+1}{x^2-4} \approx \frac{x}{x^2} = \frac{1}{x}$ As $x\to\infty$

We know $\int_3^\infty \frac{1}{x}\,dx$ diverges

$\left(\int_a^\infty \frac{1}{x^p}\right.$ converges if $p>1$, diverges if $\left.p\le 1\right)$

Comparison Theorem: we want "∞"

$$\frac{x+1}{x^2-4} \overset{?}{>} \frac{1}{x}$$

$$x^2 + x > x^2 - 4$$

$$x > -4 \quad \text{true since } x \in [3,\infty)$$

$\therefore \int_3^\infty \frac{x+1}{x^2-4}\,dx$ diverges by Comparison to $\int_3^\infty \frac{1}{x}\,dx$

2 Section 11.1 Sequences

$n=1 \quad n=2 \quad n=3 \ldots$

informally: an ordered list of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$

formally: a function whose domain is the set of nonnegative integers

Limits: if $a_n = f(n)$ for some real-valued function f, $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$

(all properties hold for sequences EXCEPT L'Hospital's Rule!!! ← cannot take derivative of a sequence)

Montonic – increasing or decreasing

show:

1) $a_{n+1} - a_n > 0$

2) $\frac{a_{n+1}}{a_n} > 1$ if a_n's positive

3) $f'(x) > 0$ if $a_n = f(n)$

Bounded : $m \le a_n \le M$

1. Find the limits of the following sequences:

(a) $a_n = \dfrac{\ln(n + e^{3n})}{n}$ $\quad f(x) = \dfrac{\ln(x+e^{3x})}{x}$ is defined

$$\lim_{x\to\infty} \frac{\ln(x+e^{3x})}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x+e^{3x}} \cdot (1+3e^{3x})}{1}$$

$$= \lim_{x\to\infty} \frac{1+3e^{3x}}{x+e^{3x}} = 3 \quad \text{(or do L'Hospital's Rule again)}$$

$\therefore$ the sequence a_n converges to $\boxed{3}$

(b) $a_n = \left(1 + \frac{3}{n}\right)^{n/2}$ $f(x) = \left(1+\frac{3}{x}\right)^{\frac{x}{2}}$ is defined

Recall from section 4.8

$$e^{\lim_{x\to\infty} \ln\left(1+\frac{3}{x}\right)^{\frac{x}{2}}}$$

$$= e^{\lim_{x\to\infty} \frac{x}{2}\ln\left(1+\frac{3}{x}\right)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{2}{x}}}$$

$\frac{\ln(1)=0}{0}$ can apply L'Hospital's Rule

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}} \cdot \frac{+3}{\cancel{x^2}}}{-\frac{2}{x^2}}} \quad + \frac{\cancel{x^2}}{2}$$

$$= e^{\lim_{x\to\infty} \frac{1}{1+\cancel{\frac{3}{x}}\to 0} \cdot \frac{3}{2}} = \boxed{e^{\frac{3}{2}}}$$

(c) $a_n = \arctan\left(\frac{n}{n+1}\right)$ Since arctan function is continuous

$$\lim_{n\to\infty} \arctan\left(\frac{n}{n+1}\right)$$

$$= \arctan\left(\lim_{n\to\infty} \frac{n}{n+1}\right) \quad \frac{n}{n+1}\to 1$$

or use L'Hospital's Rule

$$= \arctan(1)$$

arctan(1) = ? means tan(?) = 1

$$= \boxed{\frac{\pi}{4}}$$

(d) $a_n = \arctan\left(\dfrac{n^2}{n+1}\right)$*

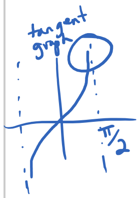
$\lim\limits_{n\to\infty} \arctan\left(\dfrac{n^2}{n+1}\right) \to \infty$

$= \arctan\left(\lim\limits_{n\to\infty} \dfrac{n^2}{n+1}\right)$

" $\arctan(\infty) = ?$ means

$\tan(?) = \infty$

$= \boxed{\dfrac{\pi}{2}}$

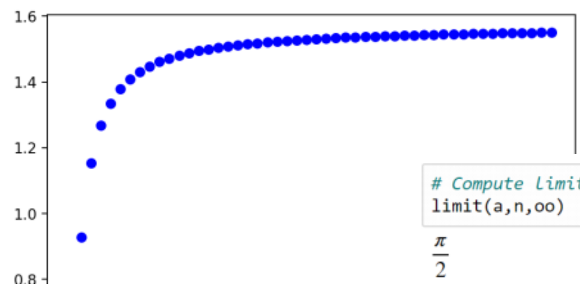



```
# List first 10 terms of sequence
n=symbols('n',positive=True,integer=True)
a=atan(n**2/(n+1))
n10=range(1,11) # NOTE range command goes from a INCLUSIVE to b EXCLUSIVE
a10=[a.subs({n:i}) for i in n10]
print(a10)  # Ran it here.  Need to convert to floating point.
a10=[a.subs({n:i}).evalf() for i in n10]  # CANNOT convert the list-must do one at a time
print(a10)  # Getting close to 1.5
```

```
[atan(1/2), atan(4/3), atan(9/4), atan(16/5), atan(25/6), atan(36/7), atan(49/8), atan(64/9), atan(81/10), atan(100/11)]
[0.463647609000806, 0.927295218001612, 1.15257199721567, 1.26791145841993, 1.33525134607403, 1.37874830955417, 1.40895889555647, 1.43108745250573, 1.44796108791700, 1.46123680002095]
```

```
# Plot first 50 terms
n50=range(1,51)
a50=[a.subs({n:i}) for i in n50]
plt.plot(n50,a50,'bo')  # Plot points using blue dot
```

Figure 1




```
# Compute limit symbolically
limit(a,n,oo)
```

$\dfrac{\pi}{2}$

(e) $a_n = \dfrac{(-1)^{n+1}}{2n+1}$

CANNOT use a real valued function here!

Look at terms: $a_n = \left\{ \underset{n=1}{\dfrac{1}{3}}, \underset{n=2}{\dfrac{-1}{5}}, \underset{n=3}{\dfrac{1}{7}}, \underset{n=4}{\dfrac{-1}{9}}, \dots \right\}$

Let $b_n = |a_n| = \dfrac{1}{2n+1}$

$\lim\limits_{n\to\infty} \dfrac{1}{2n+1} = 0$

Theorem: Since $b_n = |a_n|$ converges to 0, a_n also converges to 0.

2. Find the limit of $a_n = (\sqrt{n+1} - \sqrt{n})\sqrt{n + \frac{1}{2}}$

Conjugate

$$\lim_{n\to\infty} \sqrt{n+\tfrac{1}{2}}\,\left(\sqrt{n+1} - \sqrt{n}\right)\frac{\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+\tfrac{1}{2}}\,\left(\cancel{n}+1-\cancel{n}\right)}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+\tfrac{1}{2}}}{\sqrt{n+1}+\sqrt{n}} \approx \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \frac{\sqrt{n}}{2\sqrt{n}} = \frac{1}{2}$$

OR divide by $\sqrt{n}$

$$\lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{2n}}}{\sqrt{1+\frac{1}{n}}+\sqrt{1}} = \boxed{\frac{1}{2}}$$

($\frac{1}{2n} \to 0$, $\frac{1}{n} \to 0$)

3. Determine if the sequence $a_n = \dfrac{\ln n}{n}$ is monotonic and bounded.

Montonic: Check $f(x) = \dfrac{\ln x}{x}$

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln(x)(1)}{x^2}$$

$$= \frac{1-\ln x}{x^2} < 0 \text{ when}$$

$$1 - \ln x < 0$$
$$-\ln x < -1$$
$$\ln x > 1$$
$$x > e^1 \approx 2.718\ldots$$

So $\boxed{\text{decreasing when } n \geq 3}$

Bounded

$$0 < \frac{\ln n}{n} < \frac{\ln 3}{3} \text{ or } \frac{\ln 2}{2} < 1$$

Since all terms positive

(Not sure without a calculator which is bigger, but both are less than 1)

(Since a_n is decreasing and bounded below, we know limit exists by Montone Convergence Theorem)

4. Given the sequence defined recursively by $a_1 = 1$, $a_{n+1} = \sqrt{3 + a_n}$ is increasing and bounded above by 3, find the limit.

this implies the sequence converges to a limit

Let $a_n \to L$

We know $a_{n+1} \to L$ (same sequence, just looking at different numbers)

$$\lim_{n\to\infty} \left(a_{n+1} = \sqrt{3+a_n}\right)$$

$L = \sqrt{3+L}$ solve for L

$L^2 = 3+L$

$L^2 - L - 3 = 0$ Quadratic Formula

$$L = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2} = \boxed{\frac{1+\sqrt{13}}{2}} \text{ or } \cancel{\frac{1-\sqrt{13}}{2}}$$

$a_1 = 1$
a_n increasing
CANNOT be negative

5. Given $a_n = \dfrac{1000^n}{n!}$, show a_n is decreasing (for $n >$some N) and bounded below. What is the limit of this sequence, and why?

SKIP