MATH 152-Spring 2020
Week in Review VIII

TEXAS A&M UNIVERSITY Math Learning Center

courtesy: David J. Manuel

(covering 11.2, 11.3, and Exam II Review)

Three distinctions
1) sequence of TERMS a_n
2) sequence of PARTIAL SUMS S_N
3) sum of series S

1 Section 11.2

1) Series $S=\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$ where $S_N = \sum_{n=0}^{N} a_n$ (partial sum)

2) Geometric Series $\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + \underbrace{ar}_{\text{multiplying by } r} + ar^2 + ar^3 + \dots$

if $|r|<1$ $= \dfrac{a}{1-r}$ ← 1st term of series, ← ratio

(Diverges if $|r| \geq 1$)

3) Telescoping Series → things cancel in your partial sums
(Partial Fractions)

1. Find the sum of the series $\sum_{n=1}^{\infty} \frac{2^n + (-4)^n}{6^n}$. $= \underbrace{\sum_{n=1}^{\infty} \frac{2^n}{6^n}}_{\frac{1}{2}} + \underbrace{\sum_{n=1}^{\infty} \frac{(-4)^n}{6^n}}_{-\frac{2}{5}}$

$\frac{2}{6} + \frac{4}{36} + \frac{8}{216} + \dots$ Geometric $r = \frac{2}{6}$ Converges $|r|<1$

$\frac{-4}{6} + \frac{16}{36} + \frac{-64}{216} + \dots$ Geometric $r = \frac{-4}{6}$ Converges $|r|<1$

Geometric series converge to $\dfrac{a}{1-r}$ ← 1st term, ← ratio

$$\sum_{n=1}^{\infty} \frac{2^n}{6^n} = \frac{2/6}{1-\left(\frac{2}{6}\right)} \cdot \frac{6}{6} = \frac{2}{6-2} = \frac{2}{4} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{6^n} = \frac{-\frac{4}{6}}{1-\left(-\frac{4}{6}\right)} \cdot \frac{6}{6} = \frac{-4}{6+(+4)} = \frac{-4}{10} = \frac{-2}{5}$$

Given series sums to $\frac{1}{2} - \frac{2}{5} = \boxed{\frac{1}{10}}$

2. Find the sum of $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$.

$n(n+2)$

$n=1$: $\frac{2}{1+2}$ $n=2$: $\frac{2}{2^2+2(2)}$ $n=3$: $\frac{2}{3^2+2(3)}$

$\frac{2}{3} + \frac{2}{8} + \frac{2}{14}$

NOT SAME → NOT GEOMETRIC

Telescoping → Partial Fractions

$$n(n+2)\left(\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}\right)$$

$$2 = A(n+2) + B(n)$$

Pick $n=0$: $2 = 2A + \cancel{B(0)} \to A = 1$

Pick $n=-2$: $2 = \cancel{A(0)} + B(-2) \to B = -1$

$$\frac{2}{n(n+2)} = \frac{1}{n} + \frac{-1}{n+2}$$

$$\sum_{n=1}^{\infty} \frac{2}{n^2+2n} = \sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{-1}{n+2}\right)$$

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n} + \frac{-1}{n+2}\right)$$

$$= \underbrace{\left(\frac{1}{1} + \cancel{\frac{-1}{3}}\right)}_{n=1} + \underbrace{\left(\frac{1}{2} + \cancel{\frac{-1}{4}}\right)}_{n=2} + \underbrace{\left(\cancel{\frac{1}{3}} + \cancel{\frac{-1}{5}}\right)}_{n=3} + \underbrace{\left(\cancel{\frac{1}{4}} + \cancel{\frac{-1}{6}}\right)}_{n=4} + \cdots + \underbrace{\left(\cancel{\frac{1}{N-1}} + \frac{-1}{N+1}\right)}_{n=N-1} + \underbrace{\left(\cancel{\frac{1}{N}} + \frac{-1}{N+2}\right)}_{n=N}$$

$$S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 + \frac{1}{2} - \frac{1}{N+1}^{\nearrow 0} - \frac{1}{N+2}^{\nearrow 0} = \boxed{\frac{3}{2}}$$

Series converges to $\frac{3}{2}$

3. If $\sum_{n=1}^{N} a_n = 4 + \ln(2N) - \ln(N+1)$, what is $\sum_{n=1}^{\infty} a_n$?

Partial Sum S_N

$$\lim_{N\to\infty} 4 + \ln(2N) - \ln(N+1)$$ Properties of Logs

$$= \lim_{N\to\infty} 4 + \ln\left(\frac{2N}{N+1}\right)^{\nearrow 2} \quad \text{“}\frac{2N}{N}\text{”}$$

$$= \boxed{4 + \ln(2)}$$

Series Converges to $4 + \ln(2)$

NOTE that if $a_n \to 4 + \ln(2)$ we would say the series diverges by Test for Divergence

(If $S_N \to S$, the series converges to S
If $a_n \to S \neq 0$, the series diverges)

2 Section 11.3 Integral Test

If $a_n = f(n)$ where f is continuous, positive, and decreasing,
then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\int_0^\infty f(x)\,dx$ converges

Error Estimate Formula
If $\sum_{n=0}^{\infty} a_n$ converges by Integral Test, $S = \sum_{n=0}^{\infty} a_n$ and $S_N = \sum_{n=0}^{N} a_n$, then

$$\int_{N+1}^{\infty} f(x)\,dx \le S - S_N \le \int_N^\infty f(x)\,dx$$

1. Determine whether the series $\sum_{n=0}^{\infty} ne^{-n^2}$ is convergent or divergent.

L'Hospital's Rule

$$\lim_{n\to\infty} ne^{-n^2} = \lim_{n\to\infty} \frac{n}{e^{n^2}} = \lim_{n\to\infty} \frac{1}{2ne^{n^2}} = 0$$

Integral Test

Let $f(x) = xe^{-x^2}$

Conditions: f is continuous, positive on $(0,\infty)$, and decreasing $f'(x) = (1)e^{-x^2} + x(-2xe^{-x^2})$
$= e^{-x^2}(1-2x^2) < 0$ when $x \geq 1$

$$\int_0^\infty xe^{-x^2}\,dx$$

$$= \lim_{M\to\infty} \int_0^M \frac{\cancel{x}e^{(-x^2)}}{-2\cancel{x}}\,dx \qquad u = -x^2, \quad du = -2x\,dx$$

$$\lim_{M\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^M = \lim_{M\to\infty} -\frac{1}{2}\cancel{e^{-M^2}} + \frac{1}{2} \qquad e^{\text{"}-\infty\text{"}} \to 0$$

integral converges to $\frac{1}{2}$

$\therefore$ $\boxed{\sum_{n=0}^{\infty} ne^{-n^2} \text{ converges by Integral Test}}$ NOTE $\sum_{n=0}^{\infty} ne^{-n^2} \neq \frac{1}{2}$

$a_n = \frac{\ln(n)}{n^3} \to 0$ tells us NOTHING

2. Show the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ is convergent. Estimate the maximum possible error when using s_{10} as an approximation for the sum of the series.

$f(x) = \frac{\ln x}{x^3}$ f cts, positive on $(1,\infty)$ and decreasing (show $f' < 0$)

$\int_1^{\infty} \frac{\ln x}{x^3}\,dx$ IBP $u = \ln x$ $dv = \frac{1}{x^3} = x^{-3}\,dx$

$du = \frac{1}{x}dx$ $v = -\frac{1}{2}x^{-2}$

$$\int \frac{\ln x}{x^3}\,dx = -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int \overset{x^{-1}}{x^{-2}\cdot\frac{1}{x}}\,dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx$$

$$\int \frac{\ln x}{x^3}\,dx = \frac{-\ln x}{2x^2} + \frac{1}{2}\left(-\frac{1}{2}x^{-2}\right) = \frac{-\ln x}{2x^2} - \frac{1}{4x^2}$$

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{m\to\infty}\left(\frac{-\ln x}{2x^2} - \frac{1}{4x^2}\right)\Bigg|_1^m$$

$S - S_N \le \int_N^{\infty} f(x)\,dx$

$$= \lim_{m\to\infty} \overset{0}{\cancel{-\frac{\ln m}{2m^2}}} - \overset{0}{\cancel{\frac{1}{4m^2}}} + \overset{=0}{\frac{\ln 1}{2(1)^2}} + \frac{1}{4(1)^2} = \frac{1}{4}$$

$\int \frac{\ln x}{x^3}$ converges to $\frac{1}{4}$ so $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by Integral Test

$$S - S_{10} \le \int_{10}^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{m\to\infty}\left(\frac{-\ln x}{2x^2} - \frac{1}{4x^2}\right)\Bigg|_{10}^{m} = \lim_{m\to\infty} \overset{0}{\cancel{\frac{-\ln m}{2m^2}}} - \overset{0}{\cancel{\frac{1}{4m^2}}} + \frac{\ln(10)}{2(10)^2} + \frac{1}{4(10)^2} = \boxed{\frac{\ln(10)}{200} + \frac{1}{400}} \approx$$

3. Find the number of terms required to approximate $\sum_{n=1}^{\infty} \frac{1}{n^6}$ to within 10^{-10}.

Idea $S - S_N < 10^{-10} = \frac{1}{10^{10}}$

We know $S - S_N \le \int_N^{\infty} f(x)\,dx < \frac{1}{10^{10}}$ $f(x) = \frac{1}{x^6}$

We want $\int_N^{\infty} \overset{x^{-6}}{\frac{1}{x^6}}\,dx < \frac{1}{10^{10}}$

$$\lim_{M\to\infty} \frac{1}{-5}x^{-5}\Bigg|_N^M < \frac{1}{10^{10}}$$

$$\lim_{M\to\infty} \overset{0}{\cancel{-\frac{1}{5M^5}}} + \frac{1}{5N^5} < \frac{1}{10^{10}}$$ all positive

$$10^{10} < 5N^5$$

$$N^5 > \frac{10^{10}}{5}$$

$$\boxed{N > \sqrt[5]{\frac{10^{10}}{5}}}$$ on calc, round up

3 Exam II Review

1. Evaluate the following integrals:

$a^2 - x^2 \rightarrow$ "x" $= a\sin\theta$

if $x = \frac{\sqrt{2}}{6} \longrightarrow \sin\theta = 3\cdot\frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{2} \longrightarrow \theta = \frac{\pi}{4}$

if $x = 0 \longrightarrow \sin\theta = 3(0) = 0 \rightarrow \theta = 0$

(a) $\int_0^{\sqrt{2}/6} \sqrt{1-9x^2}\,dx$ Trig Substitution

$= \int_0^{\sqrt{2}/6} 3\sqrt{\frac{1}{9}-x^2}\,dx$ OR $\int_0^{\frac{\sqrt{2}}{6}} \sqrt{1-(3x)^2}\,dx$

Let $3x = \sin\theta$

$x = \frac{1}{3}\sin\theta$

$dx = \frac{1}{3}\cos\theta\,d\theta$

$= \int_0^{\pi/4} \sqrt{1-(\sin\theta)^2}\cdot\frac{1}{3}\cos\theta\,d\theta$

$= \frac{1}{3}\int_0^{\pi/4} \underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta} = \cos\theta}\cdot\cos\theta\,d\theta$

$= \frac{1}{3}\int_0^{\pi/4} \cos^2\theta\,d\theta$ $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$u = 2\theta$
$du = 2\,d\theta$

$= \frac{1}{3}\int_0^{\pi/4} \frac{1}{2}(1+\cos 2\theta)\,d\theta$

$= \frac{1}{6}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/4}$

$= \frac{1}{6}\left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - 0 - \frac{1}{2}\sin(0)\right)$

$= \boxed{\frac{\pi}{24} + \frac{1}{12}}$

(b) $\int \frac{(x-1)^2}{x^3+x}\,dx$ Rational Expression $\longrightarrow$ Partial Fractions

Multiply out / Factor

$\left(\frac{x^2-2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}\right)\times(x^2+1)$

$x^2-2x+1 = A(x^2+1) + (Bx+C)x$ easier to multiply out and match powers

$1x^2 - 2x + 1 = Ax^2 + A + Bx^2 + Cx$

$1 = A+B$, $B = 0$

$-2 = C$

$1 = A$

$-2\cdot\frac{1}{x^2+1}$ $\int \frac{1}{x^2+1}\,dx = \tan^{-1}x$

$\int \frac{(x-1)^2}{x^3+x}\,dx = \int\left(\frac{1}{x} + \frac{-2}{x^2+1}\right)dx$

$= \boxed{\ln|x| - 2\tan^{-1}x + C}$

$x^2-a^2 \to x = a\sec\theta$

(c) $\int \frac{\sqrt{x^2-4}}{x}\,dx$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$= \int \frac{\sqrt{(2\sec\theta)^2-4}}{\cancel{2\sec\theta}} \cdot \cancel{2\sec\theta}\tan\theta\,d\theta$

$= \int \sqrt{4\sec^2\theta-4} \cdot \tan\theta\,d\theta$

$\sqrt{4(\sec^2\theta-1)}$

$\sqrt{4\tan^2\theta} = 2\tan\theta$

$= 2\int \tan^2\theta\,d\theta$

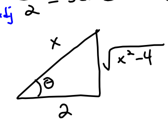
$= 2\int (\sec^2\theta - 1)\,dx$

$= 2(\tan\theta - \theta) + C$

back to x

$= \boxed{2\left(\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right) + C}$

Hyp $\frac{x}{2} = \sec\theta \to \theta = \sec^{-1}\left(\frac{x}{2}\right)$
Adj



$\tan\theta = \frac{\sqrt{x^2-4}}{2}$

(d) $\int_{-2}^{4} \frac{1}{x^2}\,dx$ Discontinuous at $x=0$

$= \int_{-2}^{0} \frac{1}{x^2}\,dx + \underbrace{\int_{0}^{4} \frac{1}{x^2}\,dx}$

$= \lim_{b\to 0^+} \int_b^4 x^{-2}\,dx$

$= \lim_{b\to 0^+} -x^{-1}\Big|_b^4$

$= \lim_{b\to 0^+} -\frac{1}{4} + \frac{1}{b} \to \infty$

$\therefore$ integral diverges

2.

(a) Evaluate $\int_1^\infty e^{-5x}\,dx$.

$u = -5x$
$du = -5\,dx$

$$= \lim_{m\to\infty} \int_1^m e^{-5x}\,dx$$

$$= \lim_{m\to\infty} -\frac{1}{5}e^{-5x}\Big|_1^m$$

$$= \lim_{m\to\infty} -\frac{1}{5}\underbrace{e^{-5m}}_{0} + \frac{1}{5}e^{-5(1)}$$

"$e^{-\infty}$"

$\therefore$ integral converges to $\boxed{\frac{1}{5}e^{-5}}$

(b) Determine whether $\int_1^\infty \frac{1}{x+e^{5x}}\,dx$ converges or diverges.

We know $\int_1^\infty \frac{1}{e^{5x}}\,dx$ converges from part (a) Use Comparison Test:

all positive

Is $\frac{1}{x+e^{5x}} \cancel{\leq} \frac{1}{e^{5x}}$ on $(1,\infty)$?

$$e^{5x} \leq x + e^{5x}$$
$$-e^{5x} \qquad -e^{5x}$$

$0 \leq x$ ✓ TRUE on $(1,\infty)$

$\therefore$ $\boxed{\int_1^\infty \frac{1}{x+e^{5x}}\,dx \text{ converges by Comparison to } \int_1^\infty e^{-5x}\,dx}$

3. Find the limit of the following or explain why they diverge:

Dominating Terms

$\frac{3n}{4n} = \frac{3}{4}$

(a) $a_n = \sqrt{\frac{3n+1}{4n+3}}$

$$\lim_{n\to\infty} \sqrt{\frac{3n+1}{4n+3}} = \sqrt{\lim_{n\to\infty} \frac{3n+1}{4n+3}}$$

$$= \boxed{\sqrt{\frac{3}{4}}} \text{ or } = \boxed{\frac{\sqrt{3}}{2}}$$

(b) $a_n = \frac{(-1)^n(2n^2+2)}{3n^2+1}$

Look at $|a_n| = \frac{2n^2+2}{3n^2+1}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n^2+2}{3n^2+1} = \frac{2}{3}$$

$(-1)^n$ alternates sign, so looks like

$-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, \ldots$

$\therefore$ $\boxed{a_n \text{ diverges by oscillation}}$

NOTE: This is a SEQUENCE not a SERIES, so

DO NOT say a_n diverges by Test for Divergence!

$\sum a_n$ diverges by Test for Divergence, NOT $\{a_n\}$

Mar 23-6:08 PM