Example 4.11, Chapter 4, Reconstruction. (Narcowich, Math 414)

Let $f(x) = 2\phi(4x)+2\phi(4x-1)+\phi(4x-2)-\phi(4x-3)$. (See figure 4.12 in the text.) Since $4 = 2^2$, $j = 2$ and $f_2 \in V_2$. The problem there was to decompose f into its components in V_0, W_0 and W_1. That is, we want $f = w_1 + w_0 + f_0$. We started with finding f_1 and w_1 in the decomposition $f_2 = f_1 + w_1$. In the example, these were found to be

$$f_1(x) = \phi(2x) \text{ and } w_1(x) = \psi(2x-1)$$

What was done next was to write $f_1 = f_0 + w_0$. Doing this we got

$$f_0(x) = \phi(x) \text{ and } w_0(x) = \psi(x)$$

The final result was this *decomposition* of f_2:

$$f = \underbrace{\phi(x)}_{f_0} + \underbrace{\psi(x)}_{w_0} + \underbrace{\psi(2x-1)}_{w_1} \tag{1}$$

We want to reverse the process and *start* with the decomposition in (1) and reconstruct f To this, we will use the method involving coefficients. The formulas that we need for this are

$$a^j_{2k} = a^{j-1}_k + b^{j-1}_k \quad a^j_{2k+1} = a^{j-1}_k - b^{j-1}_k \tag{2}$$

The decomposed form of f is given in (1). This can be written as

$$f(x) = a^0_0\phi(x) + b^0_0\psi(x) + b^1_1\psi(2x-1),$$

where.$a^0_0 = 1$, $b^0_0 = 1$ and $b^1_1 = 1$. We begin by finding the form of f at level $j = 1$. Since all of the coefficients $b^0_k = 0$, except for $k = 0$, and the same is true for a^0_k, we have for all $k \neq 0$

$$a^1_{2k} = a^0_k + b^0_k = 0 + 0 = 0 \text{ and } a^1_{2k+1} = 0 - 0 = 0.$$

For $k = 0$,

$$a^1_0 = a^0_0 + b^0_0 = 1 + 1 = 2 \text{ and } a^1_1 = a^0_0 - b^0_0 = 1 - 1 = 0.$$

This gives us

$$f(x) = 2\phi(2x) + \psi(2x-1).$$

Since there are only two non zero coefficients, which are $a_0^1 = 2$ and $b_1^1 = 1$, all the other a_k^1's and b_k^1's are 0. Using this information, first we get

$$a_0^2 = a_0^1 + b_0^1 = 2 + 0 = 0 \text{ and } a_1^2 = a_0^1 - b_0^1 = 2 - 0 = 2,$$

and then we get

$$a_2^2 = a_1^1 + b_1^1 = 0 + 1 \text{ and } a_3^2 = a_1^1 - b_1^1 = 0 - 1 = -1.$$

Of course, all other a_k^2's are 0, so

$$f(x) = 2\phi(4x) + 2\phi(4x - 1) + \phi(4x - 3) - \phi(4x - 4),$$

which was the original function. Why aren't there any wavelets from W_2? Because $f \in V_2$, f is orthogonal to all wavelets in W_2. Thus there are no components of f from W_2.