Relatioship between Fourier series for f and f'

In problem 2, HW 5 (2024), the coefficient a_0' in the series for f' has to be 0. Here's why.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)dx = \frac{1}{2\pi}\big(f(\pi) - f(-\pi)\big)$$

Since f is 2π periodic and continuous, we have that $f(\pi) = f(-\pi)$. Hence, $f(\pi) - f(-\pi) = 0$ and $a_0' = 0$. So to get a_0, you still have do the integral $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$. However, in problem 2, f is odd, so $a_0 = 0$.

As an example, consider finding the Fourier coefficients for $f(x) = x^2$, where $-\pi \le x \le \pi$. (Note that the 2π periodic extension of f is continuous and piecewise smooth, so the conditins of Theorem 1.30 apply and the series for x^2 converges uniformly.) Now, $f' = 2x$ on $-\pi < x < \pi$. It's Fourier series converges to the 2π periodic extension of f', with the extension being 0 at all multiples π. It's easy to find the FS for $2x$, which turns out to be

$$f'(x) = 2x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n}\sin(nx)$$

The formulas from the problem give, for n not equal to 0, $a_n = -(b_n')/n = -\frac{4(-1)^{n+1}}{n^2} = \frac{4(-1)^n}{n^2}$. This gives all of the a_n except $a_0 = \frac{1}{2\pi}\int_0^{\pi} x^2 dx$. Doing this integral gives $a_0 = \pi^2/3$. The series for $f(x) = x^2$ is then

$$x^2 = \pi^2/3 + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx) = \pi^2/3 - 4\cos(x) + \cos(2x) - (4/9)\cos(3x)\cdots$$

which agrees with the result in problem 1.1 in the text.

Interchanging sum and derivative. In problem 2, HW5 (2024), if f is a 2π piecewise smooth, continuous function, and it has piecewise smooth derivative f', then Fourier series for f and f' are

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

$$f'(x) = \sum_{n=1}^{\infty} a_n'\cos(nx) + b_n'\sin(nx)$$

We begin by noting that $\frac{d}{dx}\big(a_n\cos(nx) + b_n\sin(nx)\big) = nb_n\cos(nx) - na_n\sin(nx)$. Using the formulas or the coefficients a_n' and b_n' found in the problem, we have $\frac{d}{dx}\big(a_n\cos(nx) + b_n\sin(nx)\big) = a_n'\cos(nx) + b_n'\sin(nx)$. The point is that

$$f'(x) = \frac{d}{dx}\Big(\sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)\Big) = \sum_{n=1}^{\infty}\frac{d}{dx}\big(a_n\cos(nx) + b_n\sin(nx)\big).$$

Thus to obtain the series for f' it is permissible to interchange the sum and derivative in f. (Normally you can't do this.)