Relationship between Fourier series for $f$ and $f'$

In problem 2, HW 5 (2024), the coefficient $a_0'$ in the series for $f'$ has to be 0. Here's why.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) dx = \frac{1}{2\pi} (f(\pi) - f(-\pi))$$

Since $f$ is $2\pi$ periodic and continuous, we have that $f(\pi) = f(-\pi)$. Hence, $f(\pi) - f(-\pi) = 0$ and $a_0' = 0$. So to get $a_0$, you still have to do the integral $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$. However, in problem 2, $f$ is odd, so $a_0 = 0$.

As an example, consider finding the Fourier coefficients for $f(x) = x^2$, where $-\pi \leq x \leq \pi$. (Note that the $2\pi$ periodic extension of $f$ is continuous and piecewise smooth, so the conditions of Theorem 1.30 apply and the series for $x^2$ converges uniformly.) Now, $f' = 2x$ on $-\pi < x < \pi$. It's Fourier series converges to the $2\pi$ periodic extension of $f'$, with the extension being 0 at all multiples $\pi$. It's easy to find the FS for $2x$, which turns out to be

$$f'(x) = 2x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n} \sin(nx)$$

The formulas from the problem give, for $n$ not equal to 0, $a_n = -(b'_n)/n = -\frac{4(-1)^{n+1}}{n^2} = \frac{4(-1)^n}{n^2}$. This gives all of the $a_n$ except $a_0 = \frac{1}{2\pi} \int_{0}^{\pi} x^2 dx$. Doing this integral gives $a_0 = \pi^2/3$. The series for $f(x) = x^2$ is then

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) = \frac{\pi^2}{3} - 4 \cos(x) + \cos(2x) - (4/9) \cos(3x) \cdots$$

which agrees with the result in problem 1.1 in the text.

**Interchanging sum and derivative.** In problem 2, HW5 (2024), if $f$ is a $2\pi$ piecewise smooth, continuous function, and it has piecewise smooth derivative $f'$, then Fourier series for $f$ and $f'$ are

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$f'(x) = \sum_{n=1}^{\infty} a'_n \cos(nx) + b'_n \sin(nx)$$

We begin by noting that $\frac{d}{dx} (a_n \cos(nx) + b_n \sin(nx)) = nb_n \cos(nx) - na_n \sin(nx)$. Using the formulas or the coefficients $a_n'$ and $b_n'$ found in the problem, we have $\frac{d}{dx} (a_n \cos(nx) + b_n \sin(nx)) = a'_n \cos(nx) + b'_n \sin(nx)$. The point is that

$$f'(x) = \frac{d}{dx} (\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)) = \sum_{n=1}^{\infty} \frac{d}{dx} (a_n \cos(nx) + b_n \sin(nx)).$$

Thus to obtain the series for $f'$ it is permissible to interchange the sum and derivative in $f$. (Normally you can’t do this.)