Notes for April 15, 2003

The Daubechies' Wavelets. We want to find the p_k's (scaling coefficients) in the Daubechies' $N = 2$ case. This is equivalent to finding the function

$$P(z) = \frac{1}{2}\left(p_0 + p_1 z + p_2 z^2 + p_3 z^3\right). \tag{1}$$

Let us make some comments about $P(z)$ in the the general case. There are three conditions that $P(z)$ must satisfy:

1. $|P(z)|^2 + |P(-z)|^2 \equiv 1$, $|z| = 1$.
2. $P(1) = 1$.
3. $|P(e^{-it})| > 0$ for $|t| \leq \pi/2$.

Note that # 1, with $z = 1$, gives $|P(1)|^2 + |P(-1)|^2 = 1$. By # 2, $P(1) = 1$, and so $1^2 + |P(-1)|^2 = 1$, from which it follows that

$$P(-1) = 0.$$

When there are only a finite number of non-zero p_k's, P is a polynomial. Since $z = -1$ is a root of P, we see that $P(z)$ has $(z+1)^N$, for some N, as a factor; that is,

$$P(z) = (z+1)^N \widetilde{P}(z), \quad \widetilde{P}(-1) \neq 0,$$

where $\widetilde{P}(z)$ is the product of the remaining factors of P after dividing out $z + 1$ an appropriate number of times.

For the case in equation (1), P is a cubic. The values N can have as thus 1, 2, or 3. It turns out that $N = 1$ gives the Haar case ($p_0 = p_1 = 1$, $p_2 = p_3 = 0$), and $N = 3$ doesn't work. We thus assume that

$$P(z) = (z+1)^2(\alpha + \beta z),$$

where α and β are also assumed to be real. From # 2, $1 = (1+1)^2(\alpha + \beta)$, so $\alpha + \beta = 1/4$. Hence, we see that P has the form

$$P(z) = (z+1)^2(1/4 - \beta + \beta z)$$

The question remaining is, does P satisfy # 1 and # 3? To begin, we will try to find a β for which # 1 is satisfied. We do this simply by finding a value that works for $z = i$ ($|i| = 1$), and check to see if it works for all z with $|z| = 1$. We have

$$P(i) = (1+i)^2(1/4 - \beta + \beta i) = 2i(1/4 - \beta + \beta i) = -2\beta + (1/2 - 2\beta)i$$

Similarly, $P(-i) = -2\beta - (1/2 - 2\beta)i$. Consequently,

$$|P(i)|^2 + |P(-i)|^2 = 2(-2\beta)^2 + 2(1/2 - 2\beta)^2 = 16\beta^2 - 4\beta + 1/2$$

Since the left side is 1 by # 1, we end up with $16\beta^2 - 4\beta + 1/2 = 1$ or $16\beta^2 - 4\beta - 1/2 = 0$. The roots of this equation are $\beta_\pm = \frac{1 \pm \sqrt{3}}{8}$. It turns out that both values of β provide appropriate p_k's. In fact, the scaling functions they lead to are related to one another by a simple reflection of the x axis about the line $x = 3/2$. If we choose the "$-$", then

$$\begin{aligned} P(z) &= \frac{1}{8}(1+z)^2\Big((1+\sqrt{3}) + (1-\sqrt{3})z\Big) \\ &= \frac{1}{2}\Big(\underbrace{\frac{1+\sqrt{3}}{4}}_{p_0} + \underbrace{\frac{3+\sqrt{3}}{4}}_{p_1} z + \underbrace{\frac{3-\sqrt{3}}{4}}_{p_2} z^2 + \underbrace{\frac{1-\sqrt{3}}{4}}_{p_3} z^3\Big). \end{aligned}$$

These are the p_k's given in the text.

Showing that $P(z)$ satisfies # 1 in our list requires some algebra, but is not really very hard. Verifying # 3 is even easier. The only points at which $|P(z)| = 0$ are precisely the roots of P; namely, $z = -1$ (a double root) and $z = \frac{1+\sqrt{3}}{\sqrt{3}-1} \approx 3.7$. The root at $z = -1 = e^{i\pi}$ has angle $t = \pi > \pi/2$, so # 3 holds in that case. The root at $z \approx 3.7$ has $|z| > 1$, so # 3 holds there as well. Thus, for all $|t| \le \pi/2$, we have that $|P(e^{-it})| > 0$.