Exercise 1, Chapter 4. (Math 414-501, Spring 2010)

The function $f(x)$ is given by

$$f(x) = \begin{cases} -1, & 0 \le x < 1/4, \\ 4, & 1/4 \le x < 1/2, \\ 2, & 1/2 \le x < 3/4, \\ -3, & 3/4 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since f is in V_2, we can write in terms of the basis $\{\phi(2^2x - k)\}_{k=0}^{3}$ (cf. Definition 4.1 in the text):

$$f(x) = -\phi(4x) + 4\phi(4x-1) + 2\phi(4x-2) - 3\phi(4x-3).$$

The easiest way to approach decomposing f into its components along V_0, W_0, and W_1 is to use Lemma 4.10, which states that

$$\begin{aligned} \phi(2^j x) &= (\psi(2^{j-1}x) + \phi(2^{j-1}x))/2, \\ \phi(2^j x - 1) &= (\phi(2^{j-1}x) - \psi(2^{j-1}x))/2. \end{aligned}$$

Begin by getting the V_1, W_1 parts. To do this, replace the functions $\phi(4x-k)$ as follows:

$$\begin{aligned} \phi(4x) &= (\phi(2x) + \psi(2x))/2, \\ \phi(4x-1) &= (\phi(2x) - \psi(2x))/2, \\ \phi(4x-2) &= (\phi(2x-1) + \psi(2x-1))/2, \\ \phi(4x-3) &= (\phi(2x-1) - \psi(2x-1))/2 \end{aligned}$$

Using these, put f into the form

$$\begin{aligned} f(x) &= (-\frac{1}{2} + 2)\phi(2x) + (1 - \frac{3}{2})\phi(2x-1) + (-\frac{1}{2} - 2)\psi(2x) + (1 + \frac{3}{2})\psi(2x-1) \\ &= \underbrace{\frac{3}{2}\phi(2x) - \frac{1}{2}\phi(2x-1)}_{V_1} \underbrace{- \frac{5}{2}\psi(2x) + \frac{5}{2}\psi(2x-1)}_{W_1}. \end{aligned}$$

Finally, use $\phi(2x) = (\phi(x) + \psi(x))/2$ and $\phi(2x-1) = (\phi(x) - \psi(x))/2$ in the equation above to obtain

$$f(x) = \underbrace{\frac{1}{2}\phi(x)}_{V_0} + \underbrace{\psi(x)}_{W_0} \underbrace{-\frac{5}{2}\psi(2x) + \frac{5}{2}\psi(2x-1)}_{W_1},$$

which is what was asked for.