Final examination – take-home part. This part is due on Monday, 12/10/07. Each point on this part is worth 1/2% of the final exam grade. You may not get help on the test from anyone except your instructor.

1. Suppose that $f(\theta)$ is 2π-periodic function in $C^m(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2π-periodic. Here $m > 0$ is a fixed integer. Let c_k denote the he k^{th} (complex) Fourier coefficient for f, and let $c_k^{(j)}$ denote the k^{th} (complex) Fourier coefficient for $f^{(j)}$. Note: in the formulas below, let $I = [-\pi, \pi]$.

   (a) **(5 pts.)** Show that $c_k^{(j)} = (-ik)^j c_k$ and that, for $k \neq 0$, c_k satisfies the bound
   $$|c_k| \le \frac{1}{2\pi |k|^{m+1}} \|f^{(m+1)}\|_{L^1(I)}.$$

   (b) **(15 pts.)** Let $f_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the n^{th} partial sum of the Fourier series for f, $n \ge 1$. Show that there are constants C and C' such that
   $$\|f - f_n\|_{L^2(I)} \le \frac{C\|f^{(m+1)}\|_{L^1(I)}}{n^{m+\frac{1}{2}}} \text{ and } \|f - f_n\|_{C(I)} \le \frac{C'\|f^{(m+1)}\|_{L^1(I)}}{n^m}.$$

   (c) **(15 pts.)** Let $f(x)$ be the 2π-periodic function for which $f(x) = x(\pi - |x|)$ when $x \in [-\pi, \pi]$. Verify that f satisfies the conditions above with $m = 1$. With the help of (a), calculate the Fourier coefficients for f, and then plot f and f_n, for $n = 5, 10, 30$. Do this in three separate plots, one for each n.

2. **(10 pts.)** Let $\mathcal{H}$ be a *complex* Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Recall that for a selfadjoint operator L, its norm is given by $\|L\| = \sup_{\|u\|=1} |\langle Lu, u\rangle|$. Show that if L is a bounded linear operator on $\mathcal{H}$, and if $M = \sup_{\|u\|=1} |\langle Lu, u\rangle|$, then $M \le \|L\| \le 2M$, whether or not L is selfadjoint. Give an example that shows this result is *false* in a *real* Hilbert space. (There is a 2×2 counterexample!)

3. Consider the eigenvalue problem $u'' + \lambda u = 0$, $u(0) = 0$, $u(1) + u'(1) = 0$. (This problem arises in connection with solving the heat equation in a uniform bar in which the temperature is 0 at $x = 0$ and where Newton's law of cooling applies at $x = 1$.) In the following, define $Ku(x) = \int_0^1 k(x,y)u(y)dy$, where the kernel is given by

$$k(x,y) := \begin{cases} \frac{1}{2}x(2-y), & 0 \le x \le y \\ \frac{1}{2}y(2-x), & y \le x \le 1. \end{cases}$$

   (a) **(10 pts.)** Show that K is a compact, selfadjoint operator on $L^2[0,1]$. Also, show that, for $f \in C[0,1]$, the equation $u = Kf$ holds if and only if $-u'' = f$, $u(0) = 0$, $u(1) + u'(1) = 0$.

   (b) **(5 pts.)** Use part **3a** to show that the eigenfunctions ϕ_n for the eigenvalue problem are complete in $L^2[0,1]$, and that eigenvalue $\lambda_n = 1/\mu_n$, where $K\phi_n = \mu_n \phi_n$.

4. Let $\mathcal{H} := \{u \in C[0,1] : u' \in L^2[0,1] \textit{ and } u(0) = u(1) = 0\}$. With the inner product $\langle u, v\rangle_{\mathcal{H}} := \int_0^1 u'v'dx$, $\mathcal{H}$ is a real Hilbert space. In the following, define the kernel G via

$$G(x,y) := \begin{cases} x(1-y), & 0 \le x \le y, \\ y(1-x), & y \le x \le 1. \end{cases} \tag{1}$$

   (a) **(10 pts.)** Fix y. Show that $G(x,y)$ is in $\mathcal{H}$, and that for any u in $\mathcal{H}$ with a piecewise continuous derivative on $[0,1]$, $u(y) = \langle u, G(\cdot, y)\rangle_{\mathcal{H}} = \langle u, G(y, \cdot)\rangle_{\mathcal{H}}$.

   (b) **(10 pts.)** Let $X := \{x_j\}_{j=1}^N$, $0 < x_1 < x_2 < \cdots < x_N < 1$, be a set of N distinct points in $[0,1]$. Show that the set $\{G(\cdot, x_j)\}_{j=1}^N$ is linearly independent and is thus a basis for $V_X := \text{span}\{G(\cdot, x_j)\}_{j=1}^N$

   (c) **(5 pts.)** Define the $N \times N$ selfadjoint matrix $A_{j,k} := G(x_j, x_k)$. Show that A is *positive definite* – i.e., $c^T Ac > 0$ for all $0 \ne c \in \mathbb{R}^N$.

   (d) **(5 pts.)** Show that if $u \in C[0,1]$, then there is a unique $u_X \in V_X$ such that $u_X(x_j) = u(x_j)$; u_X is called an interpolant for u on X.

   (e) **(10 pts.)** Show that if $u \in \mathcal{H}$, then the interpolant u_X satisfies $\|u - u_X\|_{\mathcal{H}} = \min_{v \in V_X} \|u - v\|_{\mathcal{H}}$; that is, u_X minimizes the interpolation error (or distance of u from u_X) measured in the norm of $\mathcal{H}$.