Midterm test – take-home part. (120 points) This part is due on Monday, 10/19/15. You may get help on the test *only* from your instructor, and no one else. You *may* use other books, the web, etc. If you do so, quote the source.

1. **(15 pts.)** An $n \times n$ matrix N is said to be *normal* if and only if $N^*N = NN^*$. Show that N is diagonalizable. (Hint: follow the proof for the self-adjoint case.)

2. The Legendre polynomials are defined by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.

   (a) **(5 pts.)** Show that $P_n(-x) = (-1)^n P_n(x)$.

   (b) **(5 pts.)** Show that $P_n(x) = 2^{-n}\binom{2n}{n}x^n - 2^{-n}n\binom{2n-2}{n-2}x^{n-2}+$ lower order terms, if $n \ge 2$.

   (c) **(10 pts.)** Use the previous parts above and problem 2.2.9(b) in Keener to show that, for $n \ge 1$, $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$.

3. Suppose that $f(x)$ is 2π-periodic function in $C^{(m)}(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2π-periodic. Here $m > 0$ is a fixed integer. Let c_k denote the k^{th} (complex) Fourier coefficient for f, and let $c_k^{(j)}$ denote the k^{th} (complex) Fourier coefficient for $f^{(j)}$.

   (a) **(10 pts.)** Show that $c_k^{(j)} = (ik)^j c_k,\ j = 0, \ldots, m+1$.

   (b) **(5 pts.)** For $k \neq 0$, show that c_k satisfies the bound

   $$|c_k| \le \frac{1}{2\pi|k|^{m+1}}\|f^{(m+1)}\|_{L^1[0,2\pi]}.$$

   (c) **(10 pts.)** Let $f_n(x) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the n^{th} partial sum of the Fourier series for f, $n \ge 1$. Show that both of these hold for f.

   $$\|f - f_n\|_{L^2[0,2\pi]} \le \frac{\|f^{(m+1)}\|_{L^1[0,2\pi]}}{\sqrt{\pi}\, n^{m+\frac{1}{2}}} \text{ and } \|f - f_n\|_{C[0,2\pi]} \le \frac{\|f^{(m+1)}\|_{L^1[0,2\pi]}}{\pi\, n^m}$$

4. **(15 pts.)** Let $f(x)$ be the 2π-periodic function that equals $x^2(2\pi - x)^2$ when $x \in [0, 2\pi]$. Verify that f satisfies the conditions above with $m = 2$. Calculate the Fourier series for f''' and then use it and problem 3a to find the Fourier series for f. (You will need to find c_0 directly.)

5. Let $f \in L^2_w[0,1]$, where w is a weight function that is strictly positive and continuous on $(0,1]$, and that satisfies $\int_0^1 w(x)dx = 1$. (For example, $w(x) = \frac{1}{2}x^{-\frac{1}{2}}$ is such a function, and so is $w(x) = \frac{2}{3}x^{\frac{1}{2}}$.) Our aim is to prove the theorem below in several steps. You may assume that all functions are real valued.

   Theorem 1. *$C[0,1]$ is dense in $L^2_w[0,1]$.*

   (a) **(5 pts.)** Let $a/2 < \delta < a < 1$. Let g be continuous on $[a,1]$. Extend g to be continuous on $[0,1]$ by letting $g(x) = g(a))(x - \delta)/(a-\delta)$ on $[\delta, a]$ and 0 on $[0,\delta]$. Show that

   $$\int_0^a g(x)^2 w(x)dx \le g(a)^2 \int_\delta^a w(x)dx.$$

   (b) **(10 pts.)** Show that for $f \in L^2_w[0,1]$ and g as defined above, we have

   $$\begin{aligned}\int_0^1 (f(x)-g(x))^2 w(x)dx \quad &\le \quad 2\int_0^a f(x)^2 w(x)dx + \\ 2g(a)^2 \int_\delta^a w(x)dx \quad &+ \quad \int_a^1 (f(x)-g(x))^2 w(x)dx.\end{aligned}$$

   (c) **(10 pts.)** Show that if $f \in L^2_w[0,1]$, then $f \in L^2[a,1]$.

   (d) **(10 pts.)** Finish the proof: given $\epsilon > 0$, appropriately choose a, g, and δ, in that order, to get $\|f-g\|_{L^2_w[0,1]} < \epsilon$.

6. **(10 pts.)** Use the result above to show that the Chebyshev polynomials form a complete orthogonal set with respect to the weight function $w(x) = (1-x^2)^{-1/2}$, $-1 < x < 1$.