Final Exam

Instructions. This test is due on 12/11/2023. You may get help on the test *only* from your instructor, and no one else. You *may* use my notes, other books, the web, etc. If you do so, quote the source.

Notation. $\mathcal{H}$ denotes a separable Hilbert space; $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$; $\mathcal{C}(\mathcal{H})$, the compact operators in $\mathcal{B}(\mathcal{H})$.

1. An $n \times n$ matrix A is called a circulant if the columns of A are ciclic permutations of its first column. For example, the matrix below is a circulant. (For this matrix, $\mathbf{a} = (3\ 1\ 4\ 5)^T$.)

$$\begin{pmatrix} 3 & 5 & 4 & 1 \\ 1 & 3 & 5 & 4 \\ 4 & 1 & 3 & 5 \\ 5 & 4 & 1 & 3 \end{pmatrix}$$

   (a) **(10 pts.)** Suppose that $\mathbf{a} \in \mathbb{C}^n$ is the first column of a circulant matrix A. Let $\alpha \in \mathcal{S}_n$, where, for $j = 0, \ldots n-1$, $\alpha_j = \mathbf{a}_j$. In addition, let x, y be column vectors in $\mathbb{C}^n$, with indexes starting at $j = 0$ instead of $j = 1$. Then let $\xi, \eta \in \mathcal{S}_n$, such that $\xi_j = x_j$, $\eta_j = y_j$, for $j = 0, \ldots, n-1$. Show that $Ax = y$ is equivalent to $\eta = \alpha * \xi$.

   (b) **(10 pts.)** Use the DFT and the convolution theorem (see my notes on the DFT) to show that the eigenvalues of A are the coefficents of $\widehat{\mathbf{a}}$.

   (c) **(5 pts.)** Find the corresponding eigenvectors of A.

2. Consider the finite rank (degenerate) kernel $k(x, y) = \phi_1(x)\psi_1(y) + \phi_2(x)\psi_2(y)$, where $\phi_1(x) = 6x - 3, \phi_2(x) = 3x^2, \psi_1(y) = 1, \psi_2(y) = 8y - 6$. Let $Ku = \int_0^1 k(x, y)u(y)dy$.

   (a) **(10 pts.)** $L = I - \lambda K$ has closed range. Why? Find the values of λ for which the integral equation $u(x) - \lambda \int_0^1 k(x, y)u(y)dy = f(x)$ has a solution for every $f \in L^2[0, 1]$.

   (b) **(10 pts.)** For these values, find the resolvent kernel for $(I - \lambda K)^{-1}$.

(c) **(5 pts.)** For the values of λ for which the equation $u(x) - \lambda \int_0^1 k(x,y)u(y)dy = f(x)$ does *not* have a solution for all f, find a condition on f that guarantees a solution exists. Will the solution be unique?

3. Let $\mathcal{H} = L^2[0,1]$. Consider the boundary value problem,

$$Lu := \frac{d}{dx}\left((1+x)\frac{du}{dx}\right) = f(x),\ u(0) = 0,\ u'(1) = 0. \qquad (1)$$

   (a) **(10 pts.)** Find $G(x,y)$, the Green's function for (1).

   (b) **(5 pts.)** Let $Kf(x) = \int_0^1 G(x,y)f(y)dy$. Show that K is compact, self-adjoint, and that it has no eigenvectors corresponding to $\lambda = 0$.

   (c) **(5 pts.)** Use the spectral theorey for compact operators to show that the eigenfunctions for $\frac{d}{dx}\left((1+x)\frac{du}{dx}\right) + \lambda u = 0$, $u(0) = 0$, $u'(1) = 0$ form a complete orthogonal set. (Do *not* try to find the eigenvalues or eigenvectors. They don't have any nice analytic form.)

4. **(15 pts.)** Show that every $T \in \mathcal{D}'$ such that $x^2T = 0$ has the form $a\delta(x) + b\delta'(x)$, where a and b are constants.

5. **(15 pts.)** Suppose that $L \in \mathcal{B}(\mathcal{H})$ and that there exists $c > 0$ such that, for all f in the orthogonal complement of the null space of L, we have $\|Lf\| \geq c\|f\|$. Show that the range of L is closed.