Final Exam

Instructions. This test is due on 12/11/2023. You may get help on the test only from your instructor, and no one else. You may use my notes, other books, the web, etc. If you do so, quote the source.

Notation. $\mathcal{H}$ denotes a separable Hilbert space; $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$; $\mathcal{C}(\mathcal{H})$, the compact operators in $\mathcal{B}(\mathcal{H})$.

1. An $n \times n$ matrix $A$ is called a circulant if the columns of $A$ are cyclic permutations of its first column. For example, the matrix below is a circulant. (For this matrix, $a = (3 \ 1 \ 4 \ 5)^T$.)

$$
\begin{pmatrix}
3 & 5 & 4 & 1 \\
1 & 3 & 5 & 4 \\
4 & 1 & 3 & 5 \\
5 & 4 & 1 & 3
\end{pmatrix}
$$

(a) (10 pts.) Suppose that $a \in \mathbb{C}^n$ is the first column of a circulant matrix $A$. Let $\alpha \in S_n$, where, for $j = 0, \ldots n - 1$, $\alpha_j = a_j$. In addition, let $x, y$ be column vectors in $\mathbb{C}^n$, with indexes starting at $j = 0$ instead of $j = 1$. Then let $\xi, \eta \in S_n$, such that $\xi_j = x_j$, $\eta_j = y_j$, for $j = 0, \ldots, n - 1$. Show that $Ax = y$ is equivalent to $\eta = \alpha^* \xi$.

(b) (10 pts.) Use the DFT and the convolution theorem (see my notes on the DFT) to show that the eigenvalues of $A$ are the coefficients of $\hat{a}$.

(c) (5 pts.) Find the corresponding eigenvectors of $A$.

2. Consider the finite rank (degenerate) kernel $k(x, y) = \phi_1(x)\psi_1(y) + \phi_2(x)\psi_2(y)$, where $\phi_1(x) = 6x - 3, \phi_2(x) = 3x^2, \psi_1(y) = 1, \psi_2(y) = 8y - 6$. Let $Ku = \int_0^1 k(x, y)u(y)dy$.

(a) (10 pts.) $L = I - \lambda K$ has closed range. Why? Find the values of $\lambda$ for which the integral equation $u(x) - \lambda \int_0^1 k(x, y)u(y)dy = f(x)$ has a solution for every $f \in L^2[0, 1]$.

(b) (10 pts.) For these values, find the resolvent kernel for $(I - \lambda K)^{-1}$. 
(c) (5 pts.) For the values of $\lambda$ for which the equation $u(x) - \lambda \int_0^1 k(x, y)u(y)dy = f(x)$ does not have a solution for all $f$, find a condition on $f$ that guarantees a solution exists. Will the solution be unique?

3. Let $\mathcal{H} = L^2[0, 1]$. Consider the boundary value problem,

$$Lu := \frac{d}{dx} \left( (1 + x) \frac{du}{dx} \right) = f(x), \ u(0) = 0, \ u'(1) = 0. \quad (1)$$

(a) (10 pts.) Find $G(x, y)$, the Green’s function for (1).

(b) (5 pts.) Let $Kf(x) = \int_0^1 G(x, y)f(y)dy$. Show that $K$ is compact, self-adjoint, and that it has no eigenvectors corresponding to $\lambda = 0$.

(c) (5 pts.) Use the spectral theory for compact operators to show that the eigenfunctions for $\frac{d}{dx} \left( (1 + x) \frac{du}{dx} \right) + \lambda u = 0, \ u(0) = 0, \ u'(1) = 0$ form a complete orthogonal set. (Do not try to find the eigenvalues or eigenvectors. They don’t have any nice analytic form.)

4. (15 pts.) Show that every $T \in \mathcal{D}'$ such that $x^2T = 0$ has the form $a\delta(x) + b\delta'(x)$, where $a$ and $b$ are constants.

5. (15 pts.) Suppose that $L \in \mathcal{B}(\mathcal{H})$ and that there exists $c > 0$ such that, for all $f$ in the orthogonal complement of the null space of $L$, we have $\|Lf\| \geq c\|f\|$. Show that the range of $L$ is closed.