Coordinates and Bases

Coordinate maps. This is a brief discussion of bases and the coordinates corresponding to them. We begin with a vector space V that has the ordered basis basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $\mathbf{v} \in V$, then we can always express $\mathbf{v} \in V$ in exactly one way as a linear combination of the the vectors in B. Specifically, for any $\mathbf{v} \in V$ there are unique scalars $x_1, \ldots, x_n$ such that

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n\,. \tag{1}$$

The x_j's are the coordinates of $\mathbf{v}$ relative to B. We collect them into the coordinate vector

$$[\mathbf{v}]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Because, relative to B, the coordinates of $\mathbf{v}$ are uniquely specified, we may define a map $K_B : V \to \mathbb{C}^n$ (or $\mathbb{R}^n$) via

$$K_B(\mathbf{v}) = [\mathbf{v}]_B.$$

We will call K_B the *coordinate map* relative to B. It is easy to see that K_B is linear and has the inverse

$$K_B^{-1}(\mathbf{x}) = \mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where the x_j's are coordinates of $\mathbf{v}$.

Examples. Here are some examples. Let $V = \mathcal{P}_2$ and $B = \{1, x, x^2\}$. What is the coordinate vector $[5 + 3x - x^2]_B$? Answer:

$$[5 + 3x - x^2]_B = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}.$$

If we ask the same question for $[5 - x^2 + 3x]_B$, the answer is the *same*, because to find the coordinate vector we have to *order* the basis elements so that they are in the same order as B.

Let's turn the question around. Suppose that we are given

$$[p]_B = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},$$

then what is p? Answer: $p(x) = 3 \cdot 1 + 0 \cdot x + (-4) \cdot x^2 = 3 - 4x^2$.

Let's try another space. Let $V = \mathrm{span}\{e^t, e^{-t}\}$, which is a subspace of $C(-\infty, \infty)$. Here, we will take $B = \{e^t, e^{-t}\}$. What are coordinate vectors for $\sinh(t)$ and $\cosh(t)$? Solution: Since $\sinh(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ and $\cosh(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$, these vectors are

$$[\sinh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad [\cosh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Matrices for linear transformations. The matrix that represents a linear transformation $L : V \to W$, where V and W are vector spaces with bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $D = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, respectively, is easy to get.

$$\begin{array}{ccc} V & \xrightarrow{\quad L \quad} & W \\ {\scriptstyle K_B^{-1}}\big\uparrow & & \big\downarrow{\scriptstyle K_D} \\ \mathbb{C}^n & \xrightarrow[K_D \circ L \circ K_B^{-1}]{A_L} & \mathbb{C}^n \end{array} \tag{2}$$

Let $\mathbf{e}_k$ be the $n \times 1$ column vector having 1 as its k^{th} entry and zeros for the other entries. Recall the $A_L e_k$ is the k^{th} column of A_L, so we have that

$$A_L \mathbf{e}_k = K_D \circ L \circ K_B^{-1}(\mathbf{e}_k) = K_D(L(v_k)) = \big[L(v_k)\big]_D$$

From this we we see that

$$A_L = \big([L(v_1)]_D \ [L(v_2)]_D \ \cdots \big[L(v_n)]_D\big) = ([L(\text{B-basis}]_D)\,.$$

In words, to find A_L, we first apply L to the B basis vectors, and then find the D coordintes of the result.

A matrix example. Let $V = W = \mathcal{P}_2$, $B = D = \{1, x, x^2\}$, and $L(p) = ((1 - x^2)p')'$. To find the matrix A that represents L, we first apply L to each of the basis vectors in B.

$$L(1) = 0, \ L(x) = -2x, \ \text{and} \ L(x^2) = 2 - 6x^2.$$

Next, we find the D-basis coordinate vectors for each of these. Since $B = D$ here, we have

$$[0]_D = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [-2x]_D = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \quad [2 - 6x^2]_D = \begin{pmatrix} 2 \\ 0 \\ -6 \end{pmatrix},$$

and so the natrix that represents L is ss

$$A_L = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

Suppose that we wanted to solve the eigenvalue problem, $L(p) = \lambda p$. This equation is equivalent to the matrix equation $A_L[p]_B = \lambda[p]_B$, which is a standard eigenvalue problem. Solving that problem results in three eigenvalues, $0, -2, -6$ and three corresponding eigenvectors, $(1\ 0\ 0)^T$, $(0\ 1\ 0)^T$, $(-1/2\ 0\ 3/2)^T$. These are coordinates of the eigenvectors. The eigenvectors in polynomial form are $K_B^{-1}((1\ 0\ 0)^T) = 1$, $K_B^{-1}((0\ 1\ 0)^T) = x$, $K_B^{-1}((-1/2\ 0\ 3/2)^T) = (3x^2-1)/2$.These are the first three Legendre polynomials, $P_0 = 1$, $P_2 = x$, $P_3 = \frac{3x^2-1}{2}$.

Changing bases and coordinates. We are frequently faced with the problem of replacing a set of coordinates relative to one basis with a set for another. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $D = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be bases for an n dimensional vector space V. If $\mathbf{v} \in V$, then it has coordinate vectors relative to each basis, $\mathbf{x} = [\mathbf{v}]_B$ and $\boldsymbol{\xi} = [\mathbf{v}]_D$. This means that

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \xi_1\mathbf{w}_1 + \xi_2\mathbf{w}_2 + \cdots + \xi_n\mathbf{w}_n.$$

Suppose that we know $\mathbf{x}$ and that we want $\boldsymbol{\xi}$. First, observe that $\mathbf{v} = K_B^{-1}(\mathbf{x})$ and $\boldsymbol{\xi} = K_D(\mathbf{v})$. Putting these two together then yields

$$\boldsymbol{\xi} = K_D \circ K_B^{-1}(\mathbf{x}) = S_{B\to D}\mathbf{x}.$$

The same argument[1] that we used to get A_L, the matrix of L, we obtain

$$S_{B\to D} = K_D \circ K_B^{-1} = [[B \text{ basis }]_D], \tag{3}$$

which is the transition matrix from B coordinates to D coordinates. Of course, $S_{D\to B}$, the transition matrix from D to B coordinates, is

$$S_{D\to B} = K_B \circ K_D^{-1} = [[D \text{ basis }]_B] = S_{B\to D}^{-1}.$$

We want come back to what this means for bases. When we change bases from B to D, we are replacing every $\mathbf{v}_k$ with a linear combination of

[1]In fact, if $L = I$, the identity operator, then $A_I = K_D \circ I \circ K_B^{-1} = K_D \circ K_B^{-1}$. Thus the formula in (3) is in fact a special case of (2).

$\mathbf{w}_j$'s, which we can get from $[\mathbf{v}_k]_D$, the coordinates of $\mathbf{v}_k$ in the D basis. In terms of $S = S_{B\to D}$, we have

$$[\mathbf{v}_k]_D = (S_{1,k}\ S_{2,k}\ \cdots\ S_{n,k})^T$$

Consequently,

$$\mathbf{v}_k = \sum_{j=1}^{n} S_{j,k}\mathbf{w}_j = \sum_{j=1}^{n} (S^T)_{k,j}\mathbf{w}_j.$$

If we let $\mathbf{v} = (\mathbf{v}_1\ \mathbf{v}_2 \cdots\ \mathbf{v}_n)^T$ and $\mathbf{w} = (\mathbf{w}_1\ \mathbf{w}_2 \cdots\ \mathbf{w}_n)^T$, then we arrive at

$$\mathbf{v} = S^T\mathbf{w}.$$

We can use this to get the transition matrix in the following example. If $V = \mathcal{P}_2$ and $B = \{1-x, 1+x, 1-2x+x^2\}$ and $D = \{1, x, x^2\}$, then

$$\underbrace{\begin{pmatrix} 1-x \\ 1+x \\ 1-2x+x^2 \end{pmatrix}}_{\mathbf{v}} = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}}_{S^T} \underbrace{\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}}_{\mathbf{w}}.$$

From this we obtain the transition matrix

$$S = S_{B\to D} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get the transition matrix for $D \to B$, we just invert $S_{B\to D}$.

$$S_{D\to B} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Just to finish this example, we see that

$$\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 1/2 & -3/2 & 1 \end{pmatrix}}_{S^T_{D\to B}} \begin{pmatrix} 1-x \\ 1+x \\ 1-2x+x^2 \end{pmatrix}$$

QR factorization. We can use the techniques above to prove an important result that is frequently used in numerical analysis.

Proposition 0.1. *Let A be an $m \times n$ matrix, $m \geq n$, such that the columns of A are linearly independent. Then, there exists an $m \times n$ matrix Q, whose columns are orthonormal, and an $n \times n$ upper triangular matrix R, with positive diagonal entries, such that $A = QR$.*

Proof. See the paragraph in my notes on innerproduct spaces, **QR factorization**. □

As a simple example, consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The matrix Q has columns obtained by applying the Gram-Schmidt process to the columns of A. To find R see the method outlined in the notes mentioned in the proof above. Q and R are given below.

$$Q = \begin{pmatrix} \sqrt{2}/2 & \sqrt{6}/6 \\ 0 & -\sqrt{6}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} \sqrt{2} & 3\sqrt{2}/2 \\ 0 & \sqrt{6}/2 \end{pmatrix}.$$

Matrices for L in different bases. Let the bases B and D be as above, and suppose that A_L is the matrix for L relative to B and $\widetilde{A}_L$ be the one for D. We want to relate the two matrices. First, note that we have $A_L = K_B \circ L \circ K_B^{-1}$, and $\widetilde{A}_L = K_D \circ L \circ K_D^{-1}$. Since $K_B^{-1} \circ K_B = I$, the identity operator on V, we have

$$\widetilde{A}_L = \underbrace{K_D \circ K_B^{-1}}_{S_{B\to D}} \circ \underbrace{K_B \circ L \circ K_B^{-1}}_{A_L} \circ \underbrace{K_B \circ K_D^{-1}}_{S_{D\to B}} = S_{B\to D} A_L S_{D\to B}. \tag{4}$$

Freuqently, we let $S = S_{D\to B}$, so $S_{B\to D} = S^{-1}$. In this notation

$$\widetilde{A}_L = S^{-1} A_L S. \tag{5}$$

The matrices in (5) are similar. In fact, *any* matrix A represents L in some basis if and only if it is similar to A_L.

Previous: inner products and norms
Next: review of diagonalization