Diagonalization

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Eigenspaces. We will start with a linear transformation $L : V \to V$, where V is a finite dimensional vector space. A scalar λ (possibly complex) is an *eigenvalue* of L if there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $L[\mathbf{v}] = \lambda\mathbf{v}$. When $\mathbf{v}$ exists, it is called an *eigenvector* associated with λ. The span of all eigenvectors associated with λ is called the *eigenspace* of λ, which we will denote by $\mathcal{E}_\lambda$.

We can rewrite the equation $L[\mathbf{v}] = \lambda\mathbf{v}$ as $(L - \lambda I)\mathbf{v} = \mathbf{0}$. Conversely, the latter equation can be expanded out and rearranged to give the former, and so they are equivalent. The point is that the eigenspace of λ is the null space of $L - \lambda I$. In symbols, $\mathcal{E}_\lambda = \mathcal{N}(L - \lambda I)$. In addition, the dimension of $\mathcal{E}_\lambda$ is the nullity of $L - \lambda I$. This dimension is called the *geometric multiplicity* of λ and it will be denoted by γ_λ.

Diagonalization of linear transformations. The matrix A that represents the linear transformation $L : V \to V$ relative to a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ has columns that are the coordinate vectors $[L(\mathbf{v}_j)]_B$, $j = 1, \ldots, n$. We say that L is *diagonalizable* if there is a basis for V for composed of eigenvectors of L. When this happens the matrix of L in such a basis is diagonal. Conversely, if L can be represented by a diagonal matrix, then the basis in which this holds is composed of eigenvectors. We remark here, and will show below, that not every linear transformation can be diagonalized.

Diagonalizing L starts with finding a matrix A for L relative to some basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Once this is done, we find the eigenvalues and eigenvectors for A, following these steps.

1. Find the characteristic polynomial for A, $p_A(\lambda) = \det(A - \lambda I)$ and factor it:
$$p_A(\lambda) = (\lambda_1 - \lambda)^{\alpha_1} \cdots (\lambda_r - \lambda)^{\alpha_r}.$$

Here $\lambda_1, \ldots, \lambda_r$, which are the *distinct* roots of $p_A(\lambda)$, are the eigenvalues of A (and L, too). The number α_j is called the *algebraic multiplicity* of λ_j. It is just the number of times λ_j is repeated.

2. For each distinct eigenvalue λ_j find the corresponding eigenvectors that form a basis for its eigenspace $\mathcal{E}_{\lambda_j}$, which is the null space of $A - \lambda_j I$. This we do using standard row reduction methods. Since the geometric multiplicity γ_j for λ_j is the dimension of $\mathcal{E}_{\lambda_j}$, there will be exactly γ_j vectors in this basis.

3. **Theorem:** *The linear transformation L will be diagonalizable if and only if $\gamma_j = \alpha_j$ for $j = 1, \ldots, r$. If that happens, then the matrix*

$$S = [\mathcal{E}_{\lambda_1}\text{basis} \;\; \mathcal{E}_{\lambda_2}\text{basis} \;\; \cdots \;\; \mathcal{E}_{\lambda_r}\text{basis}]_{\text{B}-\text{coords}}$$

is the matrix that changes from coordinates relative to D, the basis of of eigenvectors, to coordinates relative to B. Finally, the matrix of L relative to D is the diagonal matrix $\Lambda = S^{-1}AS$.

An example. Let $V = \mathcal{P}_2$ and $L(p) = ((1-x^2)p')'$. Finding A, the matrix of L relative to B, starts with applying L to each of the polynomials in B: $L[1] = 0$, $L[x] = -2x$, and $L[x^2] = 2 - 6x^2$. The B-coordinate vectors for these are the columns $(0\ 0\ 0)^T$, $(0\ -2\ 0)^T$, and $(2\ 0\ -6)^T$, respectively. The matrix A is thus given by

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

The characteristic polynomial is $p_A(\lambda) = \det(A - \lambda I)$. The determinant is easy to find. The result is $p_A(\lambda) = (-\lambda)(-2-\lambda)(-6-\lambda)$. Thus, there are three distinct eigenvalues $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = -6$, with algebraic multiplicities $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Let's find the corresponding eigenvectors. For $\lambda_1 = 0$, we have

$$[A - 0 \cdot I \,|\, \mathbf{0}\,] = \left(\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Solving this system gives us $x_1 = t$ (non-leading variable), $x_2 = 0$, and $x_3 = 0$. The eigenspace $\mathcal{E}_{\lambda_1=0}$ is thus one dimensional, with $[1\ 0\ 0]^T$ being the only vector in the basis. Since γ_1 is the dimension of the eigenspace $\mathcal{E}_{\lambda_1=0}$, we see that $\gamma_1 = 1 = \alpha_1$. For $\lambda_2 = -2$, we have

$$[A - (-2)\cdot I \,|\, \mathbf{0}\,] = \left(\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The eigenvector $[0\ 1\ 0]^T$ is again the only vector in the basis for the eigenspace $\mathcal{E}_2$; also, $\gamma_2 = 1 = \alpha_2$. Finally, for $\lambda_3 = -6$, we have

$$[A - (-6)\cdot I \,|\, \mathbf{0}\,] = \left(\begin{array}{ccc|c} 6 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Solving this, we get the eigenvector $[-1\ 0\ 3]^T$, which again is the only vector in the basis for $\mathcal{E}_3$, and we have $\gamma_3 = 1 = \alpha_3$.

Since $\gamma_j = \alpha_j$, $j = 1, 2, 3$, the theorem above implies that A, and hence L, is diagonalizable. Using the theorem, we can put the eigenvalues and eigenvectors together to get the matrices Λ and S:

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We point out that the original L was a differential operator acting on $\mathcal{P}_2$. The eigenvalues are the same as above, of course. The eigenvectors are obtained from those of the matrix by recalling that the latter are coordinate vectors for the former. The eigenvectors are then just $\{1, x, 3x^2 - 1\}$.

A nondiagonalizable matrix Not every linear transformation is diagonalizable. Here is an example of one that it is not. Consider the matrix below:

$$A = \begin{pmatrix} 8 & 1 \\ -9 & 2 \end{pmatrix}.$$

The characteristic polynomial is $p_A(\lambda) = \lambda^2 - 10\lambda + 25 = (5-\lambda)^2$. thus, the only eigenvalue of A is $\lambda = 5$, which has algebraic multiplicity $\alpha = 2$. Let's find a basis for the null space. We have

$$[A - 5\cdot I \,|\, \mathbf{0}\,] = \left(\begin{array}{cc|c} 3 & 1 & 0 \\ -9 & -3 & 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right),$$

so there is only one eigenvector $[-1\ 3]^T$ in the basis for $\mathcal{E}_{\lambda=5}$. Hence, the geometric multiplicity is $\gamma = 1 < \alpha = 2$. It follows that A is *not* diagonalizable.

A diagonalizable matrix with repeated eigenvalues In the previous example, we had a matrix with repeated eigenvalues that *wasn't* diagonalizable. Are there any other matrices with this property? The answer is yes. Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial for A is $p_A(\lambda) = \det(A - \lambda I)$. With a little work, one can show that $p_A(\lambda) = (1-\lambda)^2(3-\lambda)$, from which it follows that the eigenvalues of A are $\lambda_1 = 1$, which is repeated ($\alpha_1 = 2$), and $\lambda_2 = 3$, which is not repeated ($\alpha_2 = 1$).

Let's find bases for the eigenspaces. For λ_1, we have

$$[A - 1 \cdot I \,|\, \mathbf{0}\,] = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

From the row-reduced form of A on the right, we see that the corresponding system has x_1 as a leading variable, and x_2 and x_3 as nonleading variables. Setting $x_2 = t_1$ and $x_3 = t_2$, the system becomes $x_1 = -t_2$, $x_2 = t_1$, and $x_3 = t_2$. Putting this in vector form gives us

$$\mathbf{x} = t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, a basis for $\mathcal{E}_{\lambda_1}$ is $\{(0\ 1\ 0)^T,\ (-1\ 0\ 1)^T\}$. Similarly, one can show that a basis for $\mathcal{E}_{\lambda_2}$ is $\{(1\ 0\ 1)^T\}$. It follows that $\gamma_1 = \dim(\mathcal{E}_{\lambda_1}) = 2 = \alpha_1$, and $\gamma_2 = \dim(\mathcal{E}_{\lambda_2}) = 1 = \alpha_2$. By the theorem above, the matrix is diagonalizable, and Λ and S are

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad S = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

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