Example Problems for Distributions

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Example 0.1. *Show that if $T \in \mathcal{D}'$ and $T' = 0$, then $T = C$, where C is a constant*

Solution. Since $T' = 0$, we have that $\langle T', \phi\rangle = 0$ for all $\phi \in \mathcal{D}$. Thus,

$$\langle T, \phi'\rangle = -\langle T', \phi\rangle = 0.$$

The equation $\langle T, \phi'\rangle = 0$ is not enough to determine T. The reason is that to know what T is, we have to have its value $\langle T, \phi\rangle$ for *every* $\phi \in \mathcal{D}$. However, we only know its value on *derivatives* of functions in $\mathcal{D}$. Unfortunately, this isn't enough, because there are functions in $\mathcal{D}$ that are *not* derivatives test functions. For example, the standard "bump" function,

$$\phi_0 = \begin{cases} e^{-(1-x^2)^{-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases},$$

is not the derivative of a test function: Any anti-derivative of ϕ_0 will never have compact support. (Why?)

To get around this problem, we have to characterize all test functions that *are* derivatives of test functions. Once we do this, we will use a trick to get $\langle T, \phi\rangle$ for all $\phi \in \mathcal{D}$. Suppose that $\chi = \phi'$ for some $\phi \in \mathcal{D}$. Since ϕ has support on a finite interval $[a, b]$, we have that $\int_{-\infty}^{\infty} \chi(x)dx = \phi(b) - \phi(a) = 0 - 0 = 0$. The converse of this is also true; namely, $\int_{-\infty}^{\infty} \chi(x)dx = 0$ implies that $\chi = \phi'$ for some (unique) ϕ. Just define ϕ to be

$$\phi(x) = \int_{-\infty}^{x} \chi(t)dt.$$

It's easy to check that $\phi \in \mathcal{D}$. If the support of $\chi = [a, b]$, then when $x < a$, $\int_{-\infty}^{x} \chi(t)dt = \int_{-\infty}^{x} 0dt = 0$. When $x > b$, $\int_{-\infty}^{x} \chi(t)dt = \int_{-\infty}^{b} \chi(t)dt + \int_{b}^{x} \chi(t)dt = \int_{-\infty}^{b} \chi(t)dt + 0 = \int_{-\infty}^{\infty} \chi(x)dx = 0$. *Hence, $\chi = \phi'$ if and only if $\int_{-\infty}^{\infty} \chi(x)dx = 0$.*

Next comes our trick. First, let $c_0 = \int_{-\infty}^{\infty} \phi_0(x)dx = \int_{-1}^{1} e^{-(1-x^2)^{-1}}dx$. By construction, $c_0^{-1} \int_{-\infty}^{\infty} \phi_0(x)dx = c_0/c_0 = 1$. Second, let ψ be an arbitrary test function and define

$$\chi(x) := \psi(x) - \left(\int_{-\infty}^{\infty} \psi(t)dt\right)\phi_0(x)/c_0. \qquad (1)$$

We have that $\chi \in \mathcal{D}$ because it is a linear combination of test functions. Third, note that

$$\int_{-\infty}^{\infty} \chi(x)dx = \int_{-\infty}^{\infty} \psi(x)dx - \left(\int_{-\infty}^{\infty} \psi(x)dx \right) \left(\int_{-\infty}^{\infty} (\phi_0(t)/c_0)dt \right)$$
$$= \int_{-\infty}^{\infty} \psi(x)dx - \left(\int_{-\infty}^{\infty} \psi(x)dx \right) \cdot 1 = 0.$$

By what we just said, $\chi = \phi'$, for some ϕ. Therefore,

$$\phi' = \psi - \langle 1, \psi \rangle \phi_0/c_0, \text{ where } \langle 1, \psi \rangle = \int_{-\infty}^{\infty} \psi(t)dt$$

and so $\psi = \phi' + \langle 1, \psi \rangle \phi_0/c_0$. Finally, apply T to both sides to get

$$\langle T, \psi \rangle = \underbrace{\langle T, \phi' \rangle}_{0} + \underbrace{\langle T, \phi_0/c_0 \rangle}_{C} \langle 1, \psi \rangle = \langle C, \psi \rangle.$$

Since ϕ_0/c_0 is a specific function, $C = \langle T, \phi_0 \rangle / c_0$ is a constant that is independent of ψ. The equation above then implies that $T = C$.

Example 0.2. *Find all $u \in \mathcal{D}'$ that solve $x^2 u' = 0$, in the sense of distributions.*

Solution. The equation $x^2 u' = 0$ implies that $0 = \langle x^2 u', \phi \rangle = \langle u, (x^2 \phi)' \rangle$. We begin by finding all $\chi \in \mathcal{D}$ such that $\chi(x) = (x^2 \phi(x))'$ for some $\phi \in \mathcal{D}$. Integrating this equation yields $x^2 \phi(x) = \int_0^x \chi(t)dt$. Since ϕ has compact support in an interval $[a, b]$, $x > b$ implies that $\int_0^\infty \chi(t)dt = \int_0^b \chi(t)dt = b^2 \phi(b) = 0$. Similarly, $\int_{-\infty}^0 \chi(t)dt = \int_a^0 \chi(t)dt = -a^2 \phi(a) = 0$. Differentiating $x^2 \phi(x) = \int_0^x \chi(t)dt$ yields $2x\phi(x) + x^2 \phi'(x) = \chi(x)$. Setting $x = 0$ then results in $\chi(0) = 0$. Putting these together, we see that χ satisfies the following (necessary) conditions:

$$\int_{-\infty}^{0} \chi(x)dx = \int_{0}^{\infty} \chi(x)dx = 0, \text{ and } \chi(0) = 0. \tag{2}$$

These are also sufficient. To see this, we must show that if $\chi \in \mathcal{D}$ satisfies (2), then

$$\phi(x) := \begin{cases} x^{-2} \int_0^x \chi(t)dt, & x \neq 0 \\ \frac{1}{2}\chi'(0), & x = 0. \end{cases} \tag{3}$$

is in $\mathcal{D}$. Because $\chi \in \mathcal{D}$, it has support in a finite interval $[a, b]$. Thus we, for $x > b$, have $\phi(x) = x^{-2} \int_0^x \chi(t)dt = x^{-2} \int_0^\infty \chi(t)dt = x^{-2} \cdot 0 = 0$. The same argument also gives $\phi(x) = 0$ for $x < a$. Thus, ϕ has compact support. All that is left to get that ϕ is in $\mathcal{D}$ is showing that $\phi \in C^\infty$. The only place where ϕ is not clearly C^∞ is at $x = 0$. In a neighborhood of $x = 0$, we use Taylor's Theorem plus remainder to represent χ:

$$\chi(x) = \underbrace{\chi(0)}_{0} + \chi'(0)x + \frac{1}{2}\chi''(0)x^2 \cdots + \frac{\chi^{(n+1)}(0)}{(n+1)!}x^{n+1} + R_{n+1}(x), \qquad (4)$$

where $R_{n+1}(x) = \int_0^x \frac{\chi^{(n+2)}(t)}{(n+1)!}(x-t)^{n+1}dt = \mathcal{O}\{x^{n+2}\}$. Integrating (4) and multiplying by x^{-2}, we see that

$$\phi(x) := x^{-2}\int_0^x \chi(t)dt = \frac{1}{2}\chi'(0) + \frac{1}{6}\chi''(0)x \cdots + \frac{\chi^{(n+1)}(0)}{(n+2)!}x^n + \mathcal{O}\{x^{n+1}\},$$

from which it follows that at $x = 0$ the n^{th} derivative of ϕ is $\phi^{(n)}(0) = \frac{\chi^{(n+1)}(0)}{(n+2)!}$. Since n is arbitrary, ϕ is infinitely differentiable at 0. Consequently, ϕ is C^∞ everywhere, and, because it has compact support, $\phi \in \mathcal{D}$.

The trick that we used above in (1) can also be used here. Let ϕ_0 be the bump function defined previously. Define $\phi_1(x) := \phi_0(x-1)$ and $\phi_2(x) := \phi_0(x+1)$. These functions have supports $[0, 2]$ and $[-2, 0]$, respectively. Thus, $\phi_1(0) = \phi_2(0) = 0$. In addition, $\int_0^\infty \phi_1(x)dx = \int_{-\infty}^\infty \phi_1(x)dx = \int_{-\infty}^\infty \phi_0(x)dx = c_0$. Similarly, $\int_{-\infty}^\infty \phi_2(x)dx = c_0$. Of course, because of the supposrts of ϕ_1, ϕ_2, we also have $\int_{-\infty}^0 \phi_1(x)dx = \int_0^\infty \phi_2(x)dx = 0$. Next, let $\psi \in \mathcal{D}$ and define

$$\begin{aligned}\chi(x) := \psi(x) - \psi(0)\big(e\phi_0(x) &- (e/2)\phi_1(x) - (e/2)\phi_2(x)\big)\\ -(\phi_1(x)/c_0)\int_0^\infty \psi(t)dt &- (\phi_2(x)/c_0)\int_{-\infty}^0 \psi(t)dt.\end{aligned}$$

It is easy to check that χ satisfies the conditions in (2). Thus, there exists $\phi \in \mathcal{D}$ such that $(x^2\phi(x))' = \chi(x)$, and so $\langle x^2u', \phi\rangle = \langle u, (x^2\phi(x))'\rangle = \langle u, \chi\rangle = 0$. Hence,

$$\begin{aligned}0 = \langle u, \psi\rangle - &\underbrace{\langle u, e\phi_0 - (e/2)\phi_1 - (e/2)\phi_2\rangle}_{a_0}\psi(0) - \underbrace{\langle u, c_0^{-1}\phi_1\rangle}_{a_1}\int_0^\infty \psi(t)dt\\ &- \underbrace{\langle u, c_0^{-1}\phi_2\rangle}_{a_2}\int_{-\infty}^0 \psi(t)dt.\end{aligned}$$

This result yields $u = a_0\delta(x) + a_1H(x) + a_2H(-x)$, where a_0, a_1, a_2 are arbitrary and $H(x)$ is the Heaviside step function.

Previous: spectral theory