Example Problems for Distributions

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Example 0.1. Show that if $T \in \mathcal{D}'$ and $T' = 0$, then $T = C$, where $C$ is a constant

Solution. Since $T' = 0$, we have that $\langle T', \phi \rangle = 0$ for all $\phi \in \mathcal{D}$. Thus,

$$\langle T, \phi' \rangle = -\langle T', \phi \rangle = 0.$$ 

The equation $\langle T, \phi' \rangle = 0$ is not enough to determine $T$. The reason is that to know what $T$ is, we have to have its value $\langle T, \phi \rangle$ for every $\phi \in \mathcal{D}$. However, we only know its value on derivatives of functions in $\mathcal{D}$. Unfortunately, this isn’t enough, because there are functions in $\mathcal{D}$ that are not derivatives test functions. For example, the standard “bump” function,

$$\phi_0 = \begin{cases} 
  e^{- (1-x^2)^{-1}} & |x| < 1 \\
  0 & |x| \geq 1 
\end{cases}$$

is not the derivative of a test function: Any anti-derivative of $\phi_0$ will never have compact support. (Why?)

To get around this problem, we have to characterize all test functions that are derivatives of test functions. Once we do this, we will use a trick to get $\langle T, \phi \rangle$ for all $\phi \in \mathcal{D}$. Suppose that $\chi = \phi'$ for some $\phi \in \mathcal{D}$. Since $\phi$ has support on a finite interval $[a,b]$, we have that $\int_{-\infty}^{\infty} \chi(x) dx = \phi(b) - \phi(a) = 0 - 0 = 0$. The converse of this is also true; namely, $\int_{-\infty}^{\infty} \chi(x) dx = 0$ implies that $\chi = \phi'$ for some (unique) $\phi$. Just define $\phi$ to be

$$\phi(x) = \int_{-\infty}^{x} \chi(t) dt.$$ 

It’s easy to check that $\phi \in \mathcal{D}$. If the support of $\chi = [a,b]$, then when $x < a$, $\int_{-\infty}^{x} \chi(t) dt = \int_{x}^{b} 0 dt = 0$. When $x > b$, $\int_{-\infty}^{x} \chi(t) dt = \int_{-\infty}^{b} \chi(t) dt + \int_{b}^{x} \chi(t) dt = \int_{-\infty}^{\infty} \chi(t) dt + 0 = \int_{-\infty}^{\infty} \chi(x) dx = 0$. Hence, $\chi = \phi'$ if and only if $\int_{-\infty}^{\infty} \chi(x) dx = 0$.

Next comes our trick. First, let $c_0 = \int_{-\infty}^{\infty} \phi_0(x) dx = \int_{-1}^{1} e^{- (1-x^2)^{-1}} dx$. By construction, $c_0^{-1} \int_{-\infty}^{\infty} \phi_0(x) dx = c_0 / c_0 = 1$. Second, let $\psi$ be an arbitrary test function and define

$$\chi(x) := \psi(x) - \left( \int_{-\infty}^{\infty} \psi(t) dt \right) \phi_0(x) / c_0.$$ (1)
We have that $\chi \in \mathcal{D}$ because it is a linear combination of test functions. Third, note that

$$\int_{-\infty}^{\infty} \chi(x)dx = \int_{-\infty}^{\infty} \psi(x)dx - \left( \int_{-\infty}^{\infty} \phi(t)dt \right) \left( \int_{-\infty}^{\infty} \psi(x)dx \right) = \int_{-\infty}^{\infty} \psi(x)dx - \int_{-\infty}^{\infty} \psi(x)dx \cdot 1 = 0.$$ 

By what we just said, $\chi = \phi'$, for some $\phi$. Therefore,

$$\phi' = \psi - \langle 1, \psi \rangle \phi_0/c_0,$$

where $\langle 1, \psi \rangle = \int_{-\infty}^{\infty} \psi(t)dt$. and so $\psi = \phi' + \langle 1, \psi \rangle \phi_0/c_0$. Finally, apply $T$ to both sides to get

$$\langle T, \psi \rangle = \langle T, \phi' \rangle + \langle T, \phi_0/c_0 \rangle \langle 1, \psi \rangle = \langle C, \psi \rangle.$$ 

Since $\phi_0/c_0$ is a specific function, $C = \langle T, \phi_0 \rangle/c_0$ is a constant that is independent of $\psi$. The equation above then implies that $T = C$.

**Example 0.2.** Find all $u \in \mathcal{D}'$ that solve $x^2 u' = 0$, in the sense of distributions.

**Solution.** The equation $x^2 u' = 0$ implies that $0 = \langle x^2 u', \phi \rangle = \langle u, (x^2 \phi)' \rangle$. We begin by finding all $\chi \in \mathcal{D}$ such that $\chi(x) = (x^2 \phi(x))'$ for some $\phi \in \mathcal{D}$. Integrating this equation yields $x^2 \phi(x) = \int_0^x \chi(t)dt$. Since $\phi$ has compact support in an interval $[a, b]$, $x > b$ implies that $\int_a^\infty \chi(t)dt = \int_b^0 \chi(t)dt = b^2 \phi(b) = 0$. Similarly, $\int_a^\infty \chi(t)dt = \int_a^0 \chi(t)dt = -a^2 \phi(a) = 0$. Differentiating $x^2 \phi(x) = \int_0^x \chi(t)dt$ yields $2x \phi(x) + x^2 \phi'(x) = \chi(x)$. Setting $x = 0$ then results in $\chi(0) = 0$. Putting these together, we see that $\chi$ satisfies the following (necessary) conditions:

$$\int_{-\infty}^{0} \chi(x)dx = \int_{0}^{\infty} \chi(x)dx = 0, \text{ and } \chi(0) = 0. \quad (2)$$

These are also sufficient. To see this, we must show that if $\chi \in \mathcal{D}$ satisfies [2], then

$$\phi(x) := \begin{cases} \frac{x^{-2}}{2} \int_0^x \chi(t)dt, & x \neq 0 \\ \frac{1}{2} \chi'(0), & x = 0 \end{cases} \quad (3)$$

is in $\mathcal{D}$. Because $\chi \in \mathcal{D}$, it has support in a finite interval $[a, b]$. Thus we, for $x > b$, have $\phi(x) = x^{-2} \int_0^x \chi(t)dt = x^{-2} \int_b^\infty \chi(t)dt = x^{-2} \cdot 0 = 0$. The same
argument also gives \( \phi(x) = 0 \) for \( x < a \). Thus, \( \phi \) has compact support. All that is left to get that \( \phi \) is in \( \mathcal{D} \) is showing that \( \phi \in C^\infty \). The only place where \( \phi \) is not clearly \( C^\infty \) is at \( x = 0 \). In a neighborhood of \( x = 0 \), we use Taylor’s Theorem plus remainder to represent \( \chi \):

\[
\chi(x) = \sum_{0}^{n} \frac{\chi^{(n+1)}(0)}{(n+1)!} x^{n+1} + R_{n+1}(x),
\]

where \( R_{n+1}(x) = \int_{0}^{x} \frac{\chi^{(n+2)}(t)}{(n+1)!} (x-t)^{n+1} \, dt = O(x^{n+2}) \). Integrating (4) and multiplying by \( x^{-2} \), we see that

\[
\phi(x) := x^{-2} \int_{0}^{x} \chi(t) \, dt = \frac{1}{2} \chi'(0) + \frac{1}{6} \chi''(0) x + \frac{\chi^{(n+1)}(0)}{(n+2)!} x^{n+1} + O(x^{n+1}),
\]

from which it follows that at \( x = 0 \) the \( n^{th} \) derivative of \( \phi \) is \( \phi^{(n)}(0) = \frac{\chi^{(n+1)}(0)}{(n+2)!} \). Since \( n \) is arbitrary, \( \phi \) is infinitely differentiable at 0. Consequently, \( \phi \) is \( C^\infty \) everywhere, and, because it has compact support, \( \phi \in \mathcal{D} \).

The trick that we used above in (4) can also be used here. Let \( \phi_{0} \) be the bump function defined previously. Define \( \phi_{1}(x) := \phi_{0}(x - 1) \) and \( \phi_{2}(x) := \phi_{0}(x + 1) \). These functions have supports \([0, 2]\) and \([-2, 0]\), respectively. Thus, \( \phi_{1}(0) = \phi_{2}(0) = 0 \). In addition, \( \int_{0}^{\infty} \phi_{1}(x) \, dx = \int_{-\infty}^{\infty} \phi_{1}(x) \, dx = \int_{-\infty}^{\infty} \phi_{0}(x) \, dx = c_{0} \). Similarly, \( \int_{-\infty}^{\infty} \phi_{2}(x) \, dx = c_{0} \). Of course, because of the supports of \( \phi_{1}, \phi_{2} \), we also have \( \int_{-\infty}^{0} \phi_{1}(x) \, dx + \int_{0}^{\infty} \phi_{2}(x) \, dx = 0 \). Next, let \( \psi \in \mathcal{D} \) and define

\[
\chi(x) := \psi(x) - \psi(0)(e\phi_{0}(x) - (e/2)\phi_{1}(x) - (e/2)\phi_{2}(x))
- (\phi_{1}(x)/c_{0}) \int_{0}^{\infty} \psi(t) \, dt - (\phi_{2}(x)/c_{0}) \int_{-\infty}^{0} \psi(t) \, dt.
\]

It is easy to check that \( \chi \) satisfies the conditions in (2). Thus, there exists \( \phi \in \mathcal{D} \) such that \( (x^{2}\phi(x))' = \chi(x) \), and so \( (x^{2}u', \phi) = (u, (x^{2}\phi(x))') = (u, \chi) = 0 \). Hence,

\[
0 = \langle u, \psi \rangle - \langle u, e\phi_{0}(x) - (e/2)\phi_{1}(x) - (e/2)\phi_{2}(x) \psi(0) - (u, c_{0}^{-1}\phi_{1}) \int_{0}^{\infty} \psi(t) \, dt
- (u, c_{0}^{-1}\phi_{2}) \int_{-\infty}^{0} \psi(t) \, dt.
\]

This result yields \( u = a_{0}\delta(x) + a_{1}H(x) + a_{2}H(-x) \), where \( a_{0}, a_{1}, a_{2} \) are arbitrary and \( H(x) \) is the Heaviside step function.