Neumann Expansions

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A Neumann expansion for a resolvent $(I - \lambda K)^{-1}$, where K is a bounded operator on a Banach space $\mathcal{X}$, is a power series representation of the resolvent:

$$(I - \lambda K)^{-1} = \sum_{n=0}^{\infty} \lambda^n K^n \tag{0.1}$$

Of course, it's obvious that this won't converge if λ is too large. Our aim here is find out the conditions under which the series is convergent to the resolvent. We will use the contraction theorem to do that.

If $f \in \mathcal{X}$, we want solve $u - \lambda K u = f$. Rearranging this equation puts it in the form $u = f + \lambda K u$. Let $M(u) = f + \lambda K u$; note that the solution $u = M(u)$ is a fixed point of M. To use the contraction mapping theorem, we need to do two things. First find a closed set $S \subset \mathcal{X}$ such that $M : S \to S$, where S should be the closure of an open set. Second, show that M is Lipschitz on S.

The map M is really defined for all f, u in $\mathcal{X}$. This allows us to experiment a bit. Since K is a bounded operator, $\|Ku\|_{\mathcal{X}} \le \|K\|_{\mathcal{X}} \|u\|_{\mathcal{X}}$, so

$$\|M(u)\|_{\mathcal{X}} \le \|f\|_{\mathcal{X}} + |\lambda| \|K\|_{\mathcal{X}} \|u\|_{\mathcal{X}}$$

We now make the additional assumptions that $\|u\|_{\mathcal{X}} \le \|f\|_{\mathcal{X}}$ and that $|\lambda| \|K\|_{\mathcal{X}} < 1$. Using this in the inequality above, and noting that dividing by $1 - |\lambda| \|K\|_{\mathcal{X}} < 1$ increases the quantity it's dividing, we have

$$\|M(u)\|_{\mathcal{X}} \le \|f\|_{\mathcal{X}} + \frac{|\lambda| \|K\|_{\mathcal{X}} \|f\|_{\mathcal{X}}}{1 - |\lambda| \|K\|_{\mathcal{X}}} = \frac{\|f\|_{\mathcal{X}}}{1 - |\lambda| \, \|K\|_{\mathcal{X}}}.$$

Fix f. Choose $\alpha < \|K\|_{\mathcal{X}}^{-1}$, and fix it. This provides us with these inequalities:

$$\|u\|_{\mathcal{X}} \le \|f\|_{\mathcal{X}} < \frac{\|f\|_{\mathcal{X}}}{1 - \alpha \, \|K\|_{\mathcal{X}}} \text{ and } \|M(u)\|_{\mathcal{X}} \le \frac{\|f\|_{\mathcal{X}}}{1 - \alpha \, \|K\|_{\mathcal{X}}}$$

The appropriate set S is than the closed ball $\overline{B}(0, \frac{\|f\|_{\mathcal{X}}}{1 - \alpha \, \|K\|_{\mathcal{X}}}) = \{v \in \mathcal{X} : \|v\|_{\mathcal{X}} \le \frac{\|f\|_{\mathcal{X}}}{1 - \alpha \, \|K\|_{\mathcal{X}}}\}$. The two inequalities above imply that $M : S \to S$.

The next item we need to address is the Lipschitz inequality for M. Note that $M(u) - M(v) = f + \lambda K u - f - \lambda K v = \lambda K (u - v)$, so that

$$\|M(u) - M(v)\|_{\mathcal{X}} = \le |\lambda| \, \|K\|_{\mathcal{X}} \, \|u - v\|_{\mathcal{X}} \le \alpha \, \|K\|_{\mathcal{X}} \, \|u - v\|_{\mathcal{X}}.$$

Hence the Lipschitz constant is $\alpha \, \|K\|_{\mathcal{X}} < 1$.

Finally, we will compute the iterates u_n. To start let $u_0 = f$. Then $u_1 = M(u_0) = f + \lambda K f = (I + \lambda K) f$, $u_2 = M(u_1) = f + \lambda K (f + \lambda K f) = f + \lambda K f + \lambda^2 K^2 f$, etc. The n^{th} iterate is $u_n = f + \lambda K f + \cdots + \lambda^n K^n f = (\sum_{j=0}^{n} \lambda^j K^j) f$. The contraction mapping theorem then implies that $u = \lim_{n \to \infty} M(u_n) = (\sum_{j=0}^{\infty} \lambda^j K^j) f$. Since f is arbitrary, we arrive at the Neumann expansion in (0.1), which converges (uniformly) as long as $|\lambda| \le \alpha < \|K\|_{\mathcal{X}}^{-1}$.