A Resolvent Example

by

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Problem. Let $k(x, y) = xy^2$, $K u(x) = \int_0^1 k(x, y)u(y)dy$, and $L u = u - \lambda K u$. Assume that $L$ has closed range.

1. Determine the values of $\lambda$ for which $L u = f$ has a solution for all $f$.
   Solve $L u = f$ for these values of $\lambda$.

2. For the remaining values of $\lambda$, find a condition on $f$ that guarantees a solution to $L u = f$. When $f$ satisfies this condition, solve $L u = f$.

Solution. (1) Because $R(L)$ is closed, the Fredholm alternative applies. We begin by finding $N(L^*)$. First, we have that $L^* = I - \lambda K^*$, where $K^* w = \int_0^1 k(y, x)w(y)dy = \int_0^1 y^2 x w(y)dy$. We want to find all $w$ for which $L^* w = w - \lambda \int_0^1 x^2 y w(y)dy = 0$. Note that $w = \lambda x^2 \int_0^1 y w(y)dy$, so $w = C x^2$.

Putting this back into the equation for $w$ yields $C x^2 = \lambda C x^2 \int_0^1 y^2 dy = C(\lambda/4) x^2$. Thus, $C = (\lambda/4) C$. If $\lambda/4 \neq 1$, then $C = 0$ and $N(L^*) = \{0\}$. Thus, if $\lambda/4 \neq 1$ — i.e., $\lambda \neq 4$, $L u = f$ has a solution for all $f \in L^2[0, 1]$.

To find $u$, note that $u - \lambda x \int_0^1 y^2 u(y)dy = f$, and so we only need to find $\int_0^1 y^2 u(y)dy$. The trick for doing this is to multiply $L u = f$ by $x^2$ and then integrate. Doing this results in $\int_0^1 y^2 u(y)dy - \frac{\lambda}{4} \int_0^1 y^2 u(y)dy = \int_0^1 y^2 f(y)dy$. From this we get $\int_0^1 y^2 u(y)dy = \frac{1}{1 - \lambda/4} \int_0^1 y^2 f(y)dy$. Finally, we arrive at

$$u(x) = f(x) + \frac{4\lambda}{4 - \lambda} x \int_0^1 y^2 f(y)dy = f(x) + \frac{4\lambda}{4 - \lambda} K f(x).$$

In operator form,

$$(I - \lambda K)^{-1} = I + \frac{4\lambda}{4 - \lambda} K.$$

The operator $(I - \lambda K)^{-1}$ is called the resolvent of $K$.

(2) When $\lambda = 4$, $N(L^*) = \text{span}\{x^2\}$. By the Fredholm alternative, $L u = f$ has a solution if and only if $\int_0^1 x^2 f(x)dx = 0$. To solve $u - 4x \int_0^1 y^2 u(y)dy = f$ for $u$, we first note that $\int_0^1 y^2 u(y)dy$ is not determined, because $\int_0^1 y^2 u(y)dy - \frac{3}{4} \int_0^1 y^2 u(y)dy = \int_0^1 y^2 f(y)dy = 0$. This really just says that we have consistency. The constant $C = \int_0^1 y^2 u(y)dy$ is thus arbitrary, and $u(x) = f(x) + C x$.

Previous: Projection Theorem, etc.
Next: compact operators