

Spectral Theory for Compact Self-Adjoint Operators

by

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Let \mathcal{H} be a separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the bounded linear operators on \mathcal{H} and the compact operators on \mathcal{H} , respectively.

1 The Resolvent Set and the Spectrum of an Operator

For $n \times n$ matrices, the spectrum is just the set of eigenvalues. The spectrum of a linear operator L is defined indirectly, as the complement of another set, the resolvent set. It is necessary to do this because on an infinite dimensional space the operator L may not have eigenvalues in the usual sense.

Definition 1.1. *Let $L \in \mathcal{B}(\mathcal{H})$. The resolvent set of L is $\rho(L) := \{\lambda \in \mathbb{C} : (L - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})\}$ ². The operator $R_L(\lambda) := (L - \lambda I)^{-1}$ is called the resolvent of L . The spectrum of L , $\sigma(L)$, is defined as the complement of the resolvent set: $\sigma(L) := \rho(L)^c$.*

This agrees with the definition of the spectrum in the matrix case, where the resolvent set comprises all complex numbers that are *not* eigenvalues. In terms of its spectrum, we will see that a compact operator behaves like a matrix, in the sense that its spectrum is the union of all of its eigenvalues and 0. We begin with the eigenspaces of a compact operator.

We start with two lemmas that we will need in the sequel. The first holds for all self-adjoint operators, including unbounded ones.

Lemma 1.2. *Let $L = L^*$ be in $\mathcal{B}(\mathcal{H})$. Then the eigenvalues of L are real and the eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. The proof is identical to the one given in the matrix case, and so we will skip it. □

The second lemma, which we proved earlier, is used throughout this section. In particular, it is used in the three propositions following it.

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² $(L - \lambda I)^{-1}$ may exist as an unbounded operator, but, for λ to be in the resolvent set, this inverse must be bounded.

Lemma 1.3. *Let $\{\phi_n\}_{n=1}^\infty$ be an o.n. set in \mathcal{H} and let $K \in \mathcal{C}(\mathcal{H})$. Then, $\lim_{n \rightarrow \infty} K\phi_n = 0$.*

Proof. See Lemma 2.4, Compact Sets and Compact Operators. □

Proposition 1.4. *If $K \in \mathcal{C}(\mathcal{H})$, then $\sigma(K)$ consists only of eigenvalues of K , together with 0.*

Proof. We will assume that $K = K^*$. The result is true for all compact operators, but the proof for the general case requires more work³. Suppose that $\lambda \in \sigma(K)$, $\lambda \neq 0$. By definition, $K - \lambda I$ is not boundedly invertible. This can happen either because there is a vector $u \in \mathcal{H}$, $u \neq 0$, such that $Ku = \lambda u$, or the range of $K - \lambda I$ is not all of \mathcal{H} , or both. If the former holds, then λ is an eigenvalue of K and we are done. So, we will suppose that the range of $K - \lambda I$ is not all of \mathcal{H} . Because K is compact, the Fredholm alternative applies to the operator⁴ $L = K - \lambda I$. Thus, $\mathcal{H} = N(L^*) \oplus R(L)$. Since, by assumption, $R(L) \neq \mathcal{H}$, there is a least one $w \in N(L^*)$, $w \neq 0$; that is, $L^*w = K^*w - \bar{\lambda}w = 0$. But $K^* = K$ and thus $Kw = \bar{\lambda}w$, which means that $\bar{\lambda}$ is an eigenvalue of K . However, all of the eigenvalues of $K = K^*$ are real. Thus $\bar{\lambda} = \lambda$, and hence λ is itself an eigenvalue.

We now turn to showing that 0 is in $\sigma(K)$. Suppose not. Then, $0 \in \rho(K)$ and $K^{-1} \in \mathcal{B}(\mathcal{H})$. Let $\{\phi_n\}_{n=1}^\infty$ be an o.n. set and let $\psi_n = K\phi_n$. Then, since $\phi_n = K^{-1}\psi_n$, we have that $\|\phi_n\| = 1 \leq \|K^{-1}\|\|\psi_n\|$. But, by Lemma 1.3, we have that $\lim_{n \rightarrow \infty} \|K\phi_n\| = \lim_{n \rightarrow \infty} \|\psi_n\| = 0$, which is a contradiction. Thus, 0 is in $\sigma(K)$. □

Proposition 1.5. *Let $K \in \mathcal{C}(\mathcal{H})$. If $\lambda \neq 0$ is an eigenvalue of K , with corresponding eigenspace \mathcal{E}_λ , then \mathcal{E}_λ is finite dimensional.*

Proof. Because $\mathcal{E}_\lambda = N(K - \lambda I)$, the eigenspace is closed. We may therefore choose an o.n. basis $\{\phi_n\}_{n=1}^N$ for \mathcal{E}_λ , using the Gram-Schmidt process if necessary. Suppose that $N = \infty$. Then we have that $K\phi_n = \lambda\phi_n$. Since the ϕ_n 's are o.n., this implies that $\|K\phi_n\| = |\lambda| \neq 0$. But, by Lemma 1.3, we have that $\lim_{n \rightarrow \infty} \|K\phi_n\| = 0$. This contradiction implies that N is finite. □

Proposition 1.6. *Let $K \in \mathcal{C}(\mathcal{H})$ be self-adjoint. Then 0 is the only possible accumulation point of the eigenvalues of K*

³See T. Kato, *Perturbation Theory for Linear Operators*, Theorem 6.26, p. 185.

⁴The notation used earlier was $L = I - \lambda K$. Because of the definitions of the spectrum and resolvent, this is inconvenient here.

Proof. Suppose not. Then we may choose a sequence of (distinct) eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda \neq 0$. Let ϕ_n be an eigenvector corresponding to λ_n , with $\|\phi_n\| = 1$. Because the eigenvalues are distinct, the set $\{\phi_n\}_{n=1}^{\infty}$ is orthonormal. As above, this implies two things: First, since $\|K\phi_n\| = |\lambda_n|$, $\lim_{n \rightarrow \infty} \|K\phi_n\| = \lim_{n \rightarrow \infty} |\lambda_n| = |\lambda|$. Second, by Lemma 1.3, $\lim_{n \rightarrow \infty} K\phi_n = 0$. Combining the two yields $\lambda = 0$, which is a contradiction. \square

We remark that the previous proposition is true for any compact operator, not just ones that are self adjoint.

2 Spectral Theory for Self-Adjoint Compact Operators

In this section we will prove that the self-adjoint compact operators have properties very similar to self-adjoint matrices. Essentially, the difference comes in there being an infinite o.n. basis of for \mathcal{H} composed of eigenvectors of the operator. This has application to eigenvalue problems associated with differential equations.

Lemma 2.1. *Let $L = L^*$ be in $\mathcal{B}(\mathcal{H})$. Then $\|L\| = \sup_{\|u\|=1} |\langle Lu, u \rangle|$.*

Proof. See problem 5, assignment 10 (2024). \square

Lemma 2.2. *Let $K \neq 0 \in \mathcal{C}(\mathcal{H})$ be self-adjoint. Then, either $\|K\|$ or $-\|K\|$ or possibly both, are eigenvalues.*

Proof. By Lemma 2.1, $\|K\| = \sup_{\|u\|=1} |\langle Ku, u \rangle|$. Thus we can choose a sequence $\{u_n\}_{n=1}^{\infty}$, $\|u_n\| = 1$, such that $\|K\| = \lim_{n \rightarrow \infty} |\langle Ku_n, u_n \rangle|$. Taking away absolute values, we see that the sequence $\langle Ku_n, u_n \rangle$ will converge to $\|K\|$, or $-\|K\|$, or may have subsequences that converge to either of these. We will assume that $\langle Ku_n, u_n \rangle$ converges to $\|K\|$. If not, reverse the sign of K . Next, note that

$$\begin{aligned} \|Ku_n - \|K\|u_n\|^2 &= \|Ku_n\|^2 - 2\|K\|\langle Ku_n, u_n \rangle + \|K\|^2 \\ &\leq \|K\|^2 - 2\|K\|\langle Ku_n, u_n \rangle + \|K\|^2 \\ &\leq 2\|K\|(\|K\| - \langle Ku_n, u_n \rangle) \end{aligned} \tag{2.1}$$

Because $\|K\| = \lim_{n \rightarrow \infty} \langle Ku_n, u_n \rangle$, we have $\lim_{n \rightarrow \infty} \|Ku_n - \|K\|u_n\| = 0$. Note that this does *not* mean that the sequence $\{u_n\}$ is convergent, only that $z_n := Ku_n - \|K\|u_n$ converges to 0. In fact, this result applies for any self-adjoint operator $L^* = L \in \mathcal{B}(\mathcal{H})$, not just self-adjoint compact operators.

We will now make use of K being compact. Since the sequence $\{u_n\}$ satisfies $\|u_n\| = 1$, it is bounded, we can extract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ for which $\{Ku_{n_k}\}$ is convergent. It follows that

$$\|K\| \lim_{k \rightarrow \infty} u_{n_k} = \lim_{k \rightarrow \infty} Ku_{n_k} - \lim_{k \rightarrow \infty} z_{n_k}.$$

Since $z_{n_k} \rightarrow 0$ and the $\lim_{k \rightarrow \infty} Ku_{n_k}$ exists, we have that $u := \lim_{k \rightarrow \infty} u_{n_k}$ exists as well, and, consequently, that $\lim_{k \rightarrow \infty} Ku_{n_k} = Ku$. Moreover, using this in the previous equation and noting that $z_{n_k} \rightarrow 0$, we have $\|K\|u = Ku + 0 = Ku$. Finally, because $\|u\| = \|\lim_{k \rightarrow \infty} u_{n_k}\| = 1$, $\|K\|$ is an eigenvalue of K , with $u \neq 0$ being an eigenvector. \square

We can obtain all of the eigenvalues in the same way as we did above. In showing this, we will simplify the notation in the discussion by assuming that the operator $K = K^*$ satisfies $\langle Kv, v \rangle \geq 0$ for all $v \in \mathcal{H}$. An operator with this property is said to be *nonnegative*. This really doesn't change the argument we will now give. We will begin with the idea of an invariant subspace:

Definition 2.3. *We say that a subspace \mathcal{U} of a Hilbert space \mathcal{H} is invariant under an operator $L \in \mathcal{B}(\mathcal{H})$ if and only if for all $v \in \mathcal{U}$, Lv is in \mathcal{U} .*

Invariance will enable us to put a self-adjoint operator in “diagonal” form. To see what we mean, let $K^* = K \in \mathcal{C}(\mathcal{H})$. Label the first n positive eigenvalues in decreasing order, $\|K\| = \lambda_1 > \lambda_2 > \dots > \lambda_n > 0$, and let M_n be the span of all of the eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$ and let M_n^\perp be its orthogonal complement in \mathcal{H} . Then we have the result below.

Lemma 2.4. *Both M_n and M_n^\perp are invariant under K .*

Proof. Any $v \in M_n$ is a linear combination of eigenvectors of K ; i.e., $v = \sum_{j=1}^n \alpha_j u_j$, where $Ku_j = \lambda_j u_j$. Hence,

$$Kv = \sum_{j=1}^n \alpha_j Ku_j = \sum_{j=1}^n \alpha_j \lambda_j u_j \in M_n,$$

so M_n is invariant under K . To see that M_n^\perp is also invariant we must show that if $w \in M_n^\perp$, then $Kw \in M_n^\perp$. Let $v \in M_n$ and $w \in M_n^\perp$, so the invariance of M_n implies that $Kv \in M_n$ and, hence, $\langle Kv, w \rangle = 0$. However, since $K = K^*$,

$$0 = \langle Kv, w \rangle = \langle v, K^*w \rangle = \langle v, Kw \rangle,$$

which gives us that $\langle v, Kw \rangle = 0$ and also that $\langle Kw, v \rangle = 0$. It follows that $Kw \in M_n^\perp$ and so M_n^\perp is invariant under K . \square

Lemma 2.5. *Let $K \neq 0 \in \mathcal{C}(\mathcal{H})$ be self-adjoint and nonnegative. If K has n positive eigenvalues $\|K\| = \lambda_1 > \lambda_2 > \dots > \lambda_n > 0$, then*

$$\lambda_{n+1} = \sup\{\langle Ku, u \rangle : u \in M_n^\perp, \|u\| = 1\} < \lambda_n, \quad (2.2)$$

where M_n is the span of all of the eigenvectors for λ_1 through λ_n .

Proof. The subspace M_n is invariant under K , and so is its orthogonal complement M_n^\perp . We now define K_{n+1} be the restriction of K to M_n^\perp : For all $w \in M_n^\perp$, $K_{n+1}w := Kw$. It is easy to see that K being compact on \mathcal{H} implies that K_{n+1} is compact on M_n^\perp . By Lemma 2.2, with K_{n+1} replacing K and M_n^\perp replacing \mathcal{H} , we have that

$$\|K_{n+1}\| = \sup\{\langle K_{n+1}w, w \rangle : w \in M_n^\perp, \|w\| = 1\} \quad (2.3)$$

is an eigenvalue of K_{n+1} , with $w \neq 0$ being a corresponding eigenvector; that is, $K_{n+1}w = \|K_{n+1}\|w$. However, since K_{n+1} is the restriction of K to M_n^\perp , we see that $K_{n+1}w = Kw = \|K_{n+1}\|w$. Consequently, $\|K_{n+1}\|$ is an eigenvalue of K as well. Let $\lambda_{n+1} := \|K_{n+1}\|$. We leave it as an exercise to show that $\lambda_{n+1} < \lambda_n$. \square

Proposition 2.6. *From among eigenvectors of K corresponding to the nonzero eigenvalues of K , one may select an orthonormal basis for $R(K)$. Moreover, if $R(K)$ is dense in \mathcal{H} , then that set forms an orthonormal basis for \mathcal{H} .*

Proof. We will use the notation introduced in Lemma 2.5. In addition, take P_n to be the orthogonal projection of \mathcal{H} onto M_n and P_n^\perp be that for M_n^\perp . Let $g = Ku \in R(K)$. Write u as $u = u_n + u_n^\perp$, where $u_n = P_n u$ and $u_n^\perp = P_n^\perp u$. Because both M_n and M_n^\perp are invariant under K , $g = g_n + g_n^\perp$, where $g_n = P_n Ku_n = Ku_n$ and, similarly, $g_n^\perp = Ku_n^\perp$. Consequently,

$$g - g_n = g_n^\perp = K(u - u_n) = Ku_n^\perp = K_{n+1}u_n^\perp, \text{ where } K_{n+1} = K|_{M_n^\perp}.$$

By (2.3) and the fact that $\lambda_{n+1} = \|K_{n+1}\|$, we have

$$\|g - g_n\| \leq \lambda_{n+1}\|u_n^\perp\| \leq \lambda_{n+1}\|u\|. \quad (2.4)$$

There are two possibilities. The first is that there are only a finite number n of nonzero eigenvalues, and $\lambda_{n+1} = 0$. This means $R(K) = M_n$ and so $g = g_n$. The second is that there are infinitely many nonzero eigenvalues. Since λ_n is a decreasing sequence bounded below, the limit $\lim_{n \rightarrow \infty} \lambda_n$ exists. Moreover, this limit is 0 because the only limit point of

the nonzero eigenvalues is 0. Finally, this and $\|g - g_n\| \leq |\lambda_{n+1}| \|u\|$ imply that

$$g = \lim_{n \rightarrow \infty} g_n = \sum_{k=1}^{\infty} \sum_{j=1}^{\dim \mathcal{E}_{\lambda_k}} \langle g, \phi_{k,j} \rangle \phi_{k,j},$$

from which the completeness of the basis for $R(K)$ follows immediately.

If we also have that $R(K)$ is dense in \mathcal{H} – i.e., $\overline{R(K)} = \mathcal{H}$ –, then, since every vector in $R(K)$ can be expressed in terms of the basis, it follows from the theory in the notes on *Orthonormal Sets* that the set is an orthonormal basis for \mathcal{H} . \square

We remark that (2.4) actually provides an estimate on the error made in approximating g by g_n .

Theorem 2.7 (Spectral Theorem). *Let $K \neq 0 \in \mathcal{C}(\mathcal{H})$ be self-adjoint. Then, from among the eigenvectors of K , including those for $\lambda = 0$, we may select an orthonormal basis for \mathcal{H} .*

Proof. After proving the Fredholm alternative – Theorem 3.1 in the notes on *Several Important Theorems* –, we mentioned that the closure of the range of K satisfies $\overline{R(K)} = N(K^*)^\perp$. Since $K = K^*$ and $N(K)$ is closed, we have that $\mathcal{H} = \overline{R(K)} \oplus N(K)$. The basis constructed in Proposition 2.6 for $R(K)$ is also an orthonormal basis for $\overline{R(K)}$. (Why?) Since $N(K) = \mathcal{E}_{\lambda=0}$, we may construct an orthonormal basis for it. Combining the two bases gives us an orthonormal basis for \mathcal{H} composed of eigenvectors of K . \square

Previous: closed range theorem

Next: example problems for distributions