Midterm

This test due Friday, 3/9/2011. You may consult any written or online source. You may *not* consult anyone, except your instructor

1. **(15 pts.)** Problem 5.4.2, p. 208.

2. **(10 pts.)** Problem 6.2.14, p. 276.

3. **(15 pts.)** Problem 6.4.4, p. 277.

4. **(15 pts.)** Problem 6.4.25, p. 279.

5. **(15 pts.)** Let $\alpha > 0$, $0 < \beta < 1$, and $\mu > 0$. Show that

$$\int_{-\infty}^{\infty} \frac{e^{-i\mu x}}{(x+i\alpha)^{\beta}} dx = 2e^{-\alpha\mu - \pi i\beta/2} \sin(\pi\beta)\mu^{\beta-1}\Gamma(1-\beta),$$

where z^{β} has $-\pi/2 < \arg(z) \le 3\pi/2$. (Hint: there is a branch cut for $(z+i\alpha)^{\beta}$ along the imaginary axis $\Im(z) = y$ starting at $y = -\alpha$ and running down to $y = -\infty$. Deform the contour to make use of the cut.)

6. Let $w = f(z)$ be analytic in a region containing the disk $|z| \le 1$, and suppose that $f(0) = 0$, $f'(0) \ne 0$. For z small enough, $f(z)$ maps this disk one-to-one and onto a region in the w plane containing a disk $|w| \le a$.

   (a) **(10 pts.)** Show that the function inverse to f, $g(w)$, is given by the contour integral

$$g(w) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z f'(z)}{f(z) - w} dz. \qquad (1)$$

   (b) **(10 pts.)** For $f(z) = (z-2)^2 - 4$ and $|w|$ small, expand the integrand in (1) in a power series in w. Calculate the first three coefficients in this series and verify that the result agrees with the coefficients in the power series for $z = 2 - \sqrt{w+4}$, where the square root uses principal branch in which $\arg(z) \in (-\pi, \pi]$.

7. **(15 pts.)** The following is a special case of the Paley-Wiener Theorem. Let $f(z)$ be an entire function that satisfies these conditions: (1) for $x \in \mathbb{R}$, $f(x) \in L^1(\mathbb{R})$; (2) there exist constants $A > 0$, $\rho > 0$, and $\delta > 0$ such that $|f(z)| \leq A(|z| + 1)^{-\delta} e^{\rho|\Im(z)|}$ for all $z \in \mathbb{C}$. Show that for all $\xi \in \mathbb{R}$ such that $|\xi| > \rho$ one has that

$$\int_{-\infty}^{\infty} f(x) e^{i\xi x} dx = 0.$$