Coordinate Vectors and Examples

Coordinate vectors. This is a brief discussion of coordinate vectors and the notation for them that I presented in class. Here is the setup for all of the problems. We begin with a vector space V that has a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. We always keep the same order for vectors in the basis. Technically, this is called an *ordered* basis. If $\mathbf{v} \in V$, then we can always express $\mathbf{v} \in V$ in exactly one way as a linear combination of the the vectors in B. Specifically, for any $\mathbf{v} \in V$ there are scalars $x_1, \ldots, x_n$ such that

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n\,.$$

The x_j's are the coordinates of $\mathbf{v}$ relative to B. We collect them into the coordinate vector

$$[\mathbf{v}]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Examples. Here are some examples. Let $V = \mathcal{P}_2$ and $B = \{1, x, x^2\}$. What is the coordinate vector $[5 + 3x - x^2]_B$? Answer:

$$[5 + 3x - x^2]_B = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}.$$

If we ask the same question for $[5 - x^2 + 3x]_B$, the answer is the *same*, because to find the coordinate vector we have to *order* the basis elements so that they are in the same order as B.

Let's turn the question around. Suppose that we are given

$$[p]_B = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},$$

then what is p? Answer: $p(x) = 3 \cdot 1 + 0 \cdot x + (-4) \cdot x^2 = 3 - 4x^2$.

Let's try another space. Let $V = \mathrm{span}\{e^t, e^{-t}\}$, which is a subspace of $C(-\infty, \infty)$. Here, we will take $B = \{e^t, e^{-t}\}$. What are coordinate vectors for $\sinh(t)$ and $\cosh(t)$? Solution: Since $\sinh(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ and $\cosh(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$, these vectors are

$$[\sinh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad [\cosh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Matrices for linear transformations. The matrix that represents a linear transformation $L : V \to W$, where V and W are vector spaces with bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, respectively, is easy to get. We derived it in class, but, since it is not *explicitly* done in the text, we'll derive it here, too.

We start with the equation $\mathbf{w} = L(\mathbf{v})$. Express $\mathbf{v}$ in terms of the basis B for V: $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$. Next, apply L to both sides of this equation and use the fact that L is linear to get

$$\begin{aligned} \mathbf{w} = L(\mathbf{v}) &= L(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n) \\ &= x_1L(\mathbf{v}_1) + x_2L(\mathbf{v}_2) + \cdots + x_nL(\mathbf{v}_n)\,. \end{aligned}$$

Now, take C coordinates of both sides of $\mathbf{w} = x_1L(\mathbf{v}_1) + x_2L(\mathbf{v}_2) + \cdots + x_nL(\mathbf{v}_n)$:

$$\begin{aligned} [\mathbf{w}]_C &= [x_1L(\mathbf{v}_1) + x_2L(\mathbf{v}_2) + \cdots + x_nL(\mathbf{v}_n)]_C \\ &= x_1[L(\mathbf{v}_1)]_C + x_2[L(\mathbf{v}_2)]_C + \cdots + x_n[L(\mathbf{v}_n)]_C \\ &= A\mathbf{x}, \end{aligned}$$

where the columns of A are the coordinate vectors $[L(\mathbf{v}_j)]_C$, $j = 1, \ldots, n$.

A matrix example. Let $V = W = \mathcal{P}_2$, $B = C = \{1, x, x^2\}$, and $L(p) = ((1 - x^2)p')'$. To find the matrix A that represents L, we first apply L to each of the basis vectors in B.

$$L(1) = 0,\ L(x) = -2x,\ \text{and}\ L(x^2) = 2 - 6x^2.$$

Next, we find the C-basis coordinate vectors for each of these. Since $B = C$ here, we have

$$[0]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [-2x]_C = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \quad [2 - 6x^2]_C = \begin{pmatrix} 2 \\ 0 \\ -6 \end{pmatrix},$$

and so the natrix that represents L is

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$