Coordinate Vectors and Examples

Coordinate vectors. This is a brief discussion of coordinate vectors and the notation for them that was presented in class. Here is the setup for all of the problems. We begin with a vector space V that has a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. We always keep the same order for vectors in the basis. Technically, this is called an *ordered* basis. If $\mathbf{v} \in V$, then we can always express $\mathbf{v} \in V$ in exactly one way as a linear combination of the the vectors in B. Specifically, for any $\mathbf{v} \in V$ there are scalars $x_1, \ldots, x_n$ such that

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n\,. \tag{1}$$

The x_j's are the coordinates of $\mathbf{v}$ relative to B. We collect them into the coordinate vector

$$[\mathbf{v}]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

We remark that the operation of finding coordinates is a *linear* one. That is, we have that

$$[\mathbf{v} + \mathbf{w}] = [\mathbf{v}] + [\mathbf{w}] \quad \text{and} \quad [c\mathbf{v}] = c[\mathbf{v}].$$

This is a very important property and allows us to deal use matrix methods in connection with solving problems in finite dimensional spaces.

Examples. Here are some examples. Let $V = \mathcal{P}_2$ and $B = \{1, x, x^2\}$. What is the coordinate vector $[5 + 3x - x^2]_B$? Answer:

$$[5 + 3x - x^2]_B = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}.$$

If we ask the same question for $[5 - x^2 + 3x]_B$, the answer is the *same*, because to find the coordinate vector we have to *order* the basis elements so that they are in the same order as B.

Let's turn the question around. Suppose that we are given

$$[p]_B = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},$$

then what is p? Answer: $p(x) = 3 \cdot 1 + 0 \cdot x + (-4) \cdot x^2 = 3 - 4x^2$.

Let's try another space. Let $V = \text{span}\{e^t, e^{-t}\}$, which is a subspace of $C(-\infty, \infty)$. Here, we will take $B = \{e^t, e^{-t}\}$. What are coordinate vectors for $\sinh(t)$ and $\cosh(t)$? Solution: Since $\sinh(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ and $\cosh(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$, these vectors are

$$[\sinh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad [\cosh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

There is an important special case. Suppose that the vector space V is an inner product space and the basis B is an orthogonal set of vectors – i.e., $\langle \mathbf{v}_k, \mathbf{v}_j \rangle = 0$ if $j \neq k$ and $\|\mathbf{v}_j\|^2 > 0$. From (1), we have

$$\begin{aligned} \langle \mathbf{v}, \mathbf{v}_j \rangle &= \langle x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n, \mathbf{v}_j \rangle \\ &= x_1\langle \mathbf{v}_1, \mathbf{v}_j \rangle + x_2\langle \mathbf{v}_2, \mathbf{v}_j \rangle + \cdots + x_n\langle \mathbf{v}_n, \mathbf{v}_j \rangle \\ &= x_j\|\mathbf{v}_j\|^2. \end{aligned}$$

This gives us the following: *Relative to an orthogonal basis, the coordinate* x_j *is given by*

$$x_j = \frac{\langle \mathbf{v}, \mathbf{v}_j \rangle}{\|\mathbf{v}_j\|^2}.$$

Matrices for linear transformations. The matrix that represents a linear transformation $L : V \to W$, where V and W are vector spaces with bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $D = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, respectively, is easy to get.

We start with the equation $\mathbf{w} = L(\mathbf{v})$. Express $\mathbf{v}$ in terms of the basis B for V: $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$. Next, apply L to both sides of this equation and use the fact that L is linear to get

$$\begin{aligned} \mathbf{w} = L(\mathbf{v}) &= L(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n) \\ &= x_1L(\mathbf{v}_1) + x_2L(\mathbf{v}_2) + \cdots + x_nL(\mathbf{v}_n)\,. \end{aligned}$$

Now, take C coordinates of both sides of $\mathbf{w} = x_1L(\mathbf{v}_1) + x_2L(\mathbf{v}_2) + \cdots + x_nL(\mathbf{v}_n)$:

$$\begin{aligned} [\mathbf{w}]_D &= [x_1L(\mathbf{v}_1) + x_2L(\mathbf{v}_2) + \cdots + x_nL(\mathbf{v}_n)]_D \\ &= x_1[L(\mathbf{v}_1)]_D + x_2[L(\mathbf{v}_2)]_D + \cdots + x_n[L(\mathbf{v}_n)]_D \\ &= A\mathbf{x}, \end{aligned}$$

where the columns of A are the coordinate vectors $[L(\mathbf{v}_j)]_D$, $j = 1, \ldots, n$.

A matrix example. Let $V = W = \mathcal{P}_2$, $B = D = \{1, x, x^2\}$, and $L(p) = ((1 - x^2)p')'$. To find the matrix A that represents L, we first apply L to each of the basis vectors in B.

$$L(1) = 0,\ L(x) = -2x,\ \text{and}\ L(x^2) = 2 - 6x^2.$$

Next, we find the D-basis coordinate vectors for each of these. Since $B = D$ here, we have

$$[0]_D = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [-2x]_D = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \quad [2 - 6x^2]_D = \begin{pmatrix} 2 \\ 0 \\ -6 \end{pmatrix},$$

and so the natrix that represents L is

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

Changing coordinates. We are frequently faced with the problem of replacing a set of coordinates relative to one basis with a set for another. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ be bases for a vector space V. If $\mathbf{v} \in V$, then it has coordinate vectors relative to each basis, $\mathbf{x} = [\mathbf{v}]_B$ and $\mathbf{x}' = [\mathbf{v}]_{B'}$. This means that

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = x'_1\mathbf{v}'_1 + x'_2\mathbf{v}'_2 + \cdots + x'_n\mathbf{v}'_n.$$

If we take coordinates relative to B on both sides, then we arrive at the chain of equations below:

$$\begin{aligned} \mathbf{x} &= x'_1[\mathbf{v}'_1]_B + x'_2[\mathbf{v}'_2]_B + \cdots + x'_n[\mathbf{v}'_n]_B \\ &= \underbrace{\big[[\mathbf{v}'_1]_B \cdots [\mathbf{v}'_n]_B\big]}_{C}\,\mathbf{x}' = C\mathbf{x}'. \end{aligned}$$

Of course, we also have $\mathbf{x}' = C^{-1}\mathbf{x}$. We point out that these matrices have the forms below:

$$C = C_{B \leftarrow B'} = \big[[B' \text{ basis }]_B\big] \quad \text{and} \quad C^{-1} = C_{B' \leftarrow B} = [[B \text{ basis }]_{B'}]\,.$$