Methods for Finding Bases

1 Bases for the subspaces of a matrix

Row-reduction methods can be used to find bases. Let us now look at an example illustrating how to obtain bases for the *row space*, *null space*, and *column space* of a matrix A. To begin, we look at an example, the matrix A on the left below. If we row reduce A, the result is U on the right.

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{1} & 2 & 0 \\ \mathbf{2} & \mathbf{4} & 2 & 4 \\ \mathbf{2} & \mathbf{1} & 5 & -2 \end{pmatrix} \Longleftrightarrow U = \begin{pmatrix} \mathbf{1} & 0 & 3 & -2 \\ 0 & \mathbf{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Let the rows of A be denoted by $\mathbf{r}_j$, $j = 1, 2, 3$, and the columns of A by $\mathbf{a}_k$, $k = 1, 2, 3, 4$. Similarly, ρ_j denotes the rows of U. (We will not need the columns of U.)

1.1 Row space

The *row* spaces of A and U are identical. This is because elementary row operations preserve the span of the rows and are themselves reversible operations. Let's see in detail how this works for A. The row operations we used to row reduce A are these.

$$\begin{array}{rccccc}
\textit{step } 1: & R_2 & = & R_2 - 2R_1 & = & \mathbf{r}_2 - 2\mathbf{r}_1 \\
 & R_3 & = & R_3 - 2R_1 & = & \mathbf{r}_3 - 2\mathbf{r}_1 \\
\textit{step } 2: & R_3 & = & R_3 + \frac{1}{2}R_2 & = & \mathbf{0} \\
\textit{step } 3: & R_2 & = & \frac{1}{2}R_2 & = & \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_1 \\
\textit{step } 4: & R_1 & = & R_1 - R_2 & = & 2\mathbf{r}_1 - \frac{1}{2}\mathbf{r}_2
\end{array}$$

Inspecting these row operations shows that the rows of U satisfy

$$\rho_1 = 2\mathbf{r}_1 - \frac{1}{2}\mathbf{r}_2 \quad \rho_2 = \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_1 \quad \rho_3 = \mathbf{0}.$$

It's not hard to run the row operations backwards to get the rows of A in terms of those of U.

$$\mathbf{r}_1 = \rho_1 + \rho_2 \quad \mathbf{r}_2 = 2\rho_1 + 4\rho_2 \quad \mathbf{r}_3 = 2\rho_1 + \rho_2.$$

Thus we see that the nonzero rows of U span the row space of A.

They are also linearly independent. To test this, we begin with the equation

$$c_1\rho_1 + c_2\rho_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

Inserting the rows in the last equation we get

$$\begin{pmatrix} c_1 & c_2 & 3c_1 - c_2 & -2c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}.$$

This gives us $c_1 = c_2 = 0$, so the rows are linearly independent. Since they also span the row space of A, they form a basis for the row space of A. This is a general fact:

Theorem 1.1 *The nonzero rows in U, the reduced row echelon form of a matrix A, comprise a basis for the row space of A.*

1.1.1 Rank

The *rank* of a matrix A is defined to be the dimension of the row space. Since the dimension of a space is the number of vectors in a basis, the rank of a matrix is just the number of nonzero rows in the reduced row echelon form U. That number also equals the number of leading entries in the U, which in turn agrees with the number of *leading variables* in the corresponding homogeneous system.

Corollary 1.2 *Let U be the reduced row echelon form of a matrix A. Then, the number of nonzero zero rows in U, the number of leading entries in U, and the number of leading variables in the corresponding homogeneous sustem $A\mathbf{x} = \mathbf{0}$ all equal rank(A).*

As an example, consider the matrices A and U in (1). U has two nonzero rows, so $\text{rank}(A) = 2$. This agrees with the number of leading entries, which are $U_{1,1}$ and $U_{2,1}$. (These are in boldface in (1)). Finally, the leading variables for the homogeneous system $A\mathbf{x} = \mathbf{0}$ are x_1 and x_2, again there are two.

1.2 Null space

We recall that the null spaces of A and U are identical, because row operations don't change the solutions to the homogeneous equations involved. Let's look at an example where A and U are the matrices in (1). The equations we get from finding the null space of U – *i.e.*, solving $U\mathbf{x} = \mathbf{0}$ – are

$$\begin{aligned} x_1 + 3x_3 - 2x_4 &= 0 \\ x_2 - x_3 + 2x_4 &= 0. \end{aligned}$$

The leading variables correspond to the columns containing the leading entries, which are in boldface in U in (1); these are the variables x_1 and x_2. The remaining variables, x_3 and x_4, are free (nonleading) variables. To emphasize this, we assign them new labels, $x_3 = t_1$ and $x_4 = t_2$. (In class we frequently used α and β; this isn't convenient here.) Solving the system obtained above, we get

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\mathbf{n}_1} + t_2 \underbrace{\begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{n}_2} = t_1\mathbf{n}_1 + t_2\mathbf{n}_2 \,. \qquad (2)$$

From this equation, it is easy to show that the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ form a basis for the null space. Notice that we can get these vectors by solving $U\mathbf{x} = \mathbf{0}$ first with $t_1 = 1, t_2 = 0$ and then with $t_1 = 0, t_2 = 1$.

This works in the general case as well: *The usual procedure for solving a homogeneous system $A\mathbf{x} = \mathbf{0}$ results in a basis for the null space.* More precisely, to find a basis for the null space, begin by identifying the leading variables $x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_r}$, where r is the number of leading variables, and the free variables $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$. For the free variables, let $t_j = x_{f_j}$. Find the $n - r$ solutions to $U\mathbf{x} = \mathbf{0}$ corresponding to the free-variable choices $t_1 = 1, t_2 = 0, \ldots, t_{n-r} = 0$, $t_1 = 1, t_2 = 0, \ldots, t_{n-r} = 0$, $\ldots$, $t_1 = 0, t_2 = 0, \ldots, t_{n-r} = 1$. Call these vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{n-r}$. *The set $\{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{n-r}\}$ is a basis for the null space of A (and, of course, U).*

1.2.1 Nullity

The *nullity* of a matrix A is defined to be the dimension of the null space, $N(A)$. From our discussion above, $\text{nullity}(A)$ is just the number of *free variables*. However, the number of free variables plus the number of leading variables is the total number of variables involved, which equals the number of columns of A. Consequently, we have proved the following result:

Theorem 1.3 *Let $A \in \mathbb{R}^{m\times n}$. Then,*

$$\text{rank}(A) + \text{nullity}(A) = n = \#\ \textit{of columns of } A.$$

1.3 Column space

We now turn to finding a basis for the column space of the a matrix A. To begin, consider A and U in (1). Equation (2) above gives vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ that form a basis for $N(A)$; they satisfy $A\mathbf{n}_1 = \mathbf{0}$ and $A\mathbf{n}_2 = \mathbf{0}$. Writing these two vector equations using the "basic matrix trick" gives us:

$$-3\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \quad \text{and} \quad 2\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}\,.$$

We can use these to solve for the free columns in terms of the leading columns,

$$\mathbf{a}_3 = 3\mathbf{a}_1 - \mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_4 = -2\mathbf{a}_1 + 2\mathbf{a}_2\,.$$

Thus the column space is spanned by the set $\{\mathbf{a}_1, \mathbf{a}_2\}$. ($\mathbf{a}_1$ and $\mathbf{a}_2$ are in boldface in our matrix A above in (1).) This set is also linearly independent because the equation

$$0 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + 0\mathbf{a}_3 + 0\mathbf{a}_4 = A\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix}$$

implies that $\begin{pmatrix} x_1 & x_2 & 0 & 0 \end{pmatrix}^T$ is in the null space of A. Matching this vector with the general form of a vector in the null space shows that the corresponding t_1 and t_2 are 0, and therefore so are x_1 and x_2. It follows that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent. Since it spans the columns as well, it is a basis for the *column space* of A. Note that these columns correspond to the *leading* variables in the problems, x_1 and x_2. This is no accident. The argument that we used can be employed to show that this is true in general:

Theorem 1.4 *Let $A \in \mathbb{R}^{m\times n}$. The columns of A that correspond to the leading variables in the associated homogeneous problem, $U\mathbf{x} = \mathbf{0}$, form a basis for the column space of A. In addition, the dimension of the column space of A is* $\operatorname{rank}(A)$*.*

2 Another matrix example

Let's do another example. Consider the matrix A and the matrix U, its row reduced form, shown below.

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 & 3 \\ -2 & -1 & 2 & 1 & 1 \\ -1 & 2 & 1 & 3 & 4 \\ 0 & 5 & 0 & 5 & 7 \end{pmatrix} \Longleftrightarrow U = \begin{pmatrix} 1 & 0 & -1 & -1 & -\frac{6}{5} \\ 0 & 1 & 0 & 1 & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From U, we can read off a basis for the row space,

$$\left\{ \begin{pmatrix} 1 & 0 & -1 & -1 & -\frac{6}{5} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & \frac{7}{5} \end{pmatrix} \right\}.$$

Again, from U we see that the leading variables are x_1 and x_2, so the leading columns in A are $\mathbf{a}_1$ and $\mathbf{a}_2$. Thus, a basis for the column space is the set

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \\ 5 \end{pmatrix} \right\}.$$

To get a basis for the null space, note that the free variables are x_3 through x_5. Let $t_1 = x_3$, etc. The system corresponding to $U\mathbf{x} = \mathbf{0}$ then has the form

$$\begin{aligned} x_1 - t_1 - t_2 - \frac{6}{5}t_3 &= 0 \\ x_2 + t_2 + \frac{7}{5}t_3 &= 0\,. \end{aligned}$$

To get $\mathbf{n}_1$, set $t_1 = 1$, $t_2 = t_3 = 0$ and solve for x_1 and x_2. This gives us $\mathbf{n}_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix}^T$. For $\mathbf{n}_2$, set $t_1 = 0$, $t_2 = 1$, $t_3 = 0$, in the system above; the result is $\mathbf{n}_2 = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \end{pmatrix}^T$. Last, set $t_1 = 0$, $t_2 = 0$, $t_3 = 1$ to get $\mathbf{n}_3 = \begin{pmatrix} \frac{6}{5} & -\frac{7}{5} & 0 & 0 & 1 \end{pmatrix}^T$. The basis for the null space is thus

$$\left\{ \mathbf{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{n}_3 = \begin{pmatrix} \frac{6}{5} \\ -\frac{7}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We want to make a few remarks on this example, concerning the dimensions of the spaces involved. The common dimension of both the row space and the column space is $\mathrm{rank}(A) = 2$, which is also the number of leading variables. The dimension of the null space is the nullity of A. Here, $\mathrm{nullity}(A) = 3$. Thus, in this case we have verified that

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = 5,$$

the number of columns of A.

3 Rank and Solutions to Systems of Equations

One of the most important applications of the rank of a matrix is determining whether a system of equations is inconsistent (no solutions) or consistent, and if it is consistent, whether it has one solution or infinitely many solutions. Consider this system S of linear equations:

$$S: \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{cases}$$

Put S into augmented matrix form $[A|\mathbf{b}]$, where A is the $m \times n$ coefficient matrix for S, $\mathbf{x}$ is the $n \times 1$ vector of unknowns, and $\mathbf{b}$ is the $m \times 1$ vector of b_j's. Then we have the following theorem, which is a tool that will help in deciding whther S has none, one or many solutions.

Theorem 3.1 *Consider the system S with coefficient matrix A and augmented matrix $[A|\mathbf{b}]$. As above, the sizes of $\mathbf{b}$, A, and $[A|\mathbf{b}]$ are $m \times 1$, $m \times n$, and $m \times (n+1)$, respectively; in addition, n is the number of unknowns. We have these possibilities:*

1. *S is inconsistent if and only if* $\text{rank}[A] < \text{rank}[A|\mathbf{b}]$.
2. *S has a unique solution if and only if* $\text{rank}[A] = \text{rank}[A|\mathbf{b}] = n$.
3. *S has infinitely many solutions if and only if* $\text{rank}[A] = \text{rank}[A|\mathbf{b}] < n$.

We will skip the proof, which essentially involves what we have already learned about row reduction and linear systems. Instead, we will do a few examples.

Example 3.2 *Let* $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}$ *and* $\mathbf{b} = (1 \; -3 \;\; 4)^T$. *Determine whether the system $A\mathbf{x} = \mathbf{b}$ is consistent.*

Form the augnented matrix

$$[A|\mathbf{b}] = \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 0 & -3 \\ 2 & 3 & 4 \end{array}\right)$$

and find its reduced row echelon form, U:

$$[A|\mathbf{b}] \Longleftrightarrow U = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

Because of the way the row reduction process is done, the first two columns of U are the reduced row echelon form of A. that is,

$$A \Longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

From the equations above, we see that $\text{rank}(A) = 2 < \text{rank}[A|\mathbf{b}] = 3$. By part 1 of Theorem 3.1, the system is inconsistent.

Example 3.3 *Let* $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \end{pmatrix}$ *and* $\mathbf{b} = (3\ \ -1)^T$. *Determine whether the system* $A\mathbf{x} = \mathbf{b}$ *is consistent.*

The augmented form of the system and its reduced row echelon form are given below.

$$[A|\mathbf{b}] = \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 3 \\ 2 & 2 & -3 & 1 & -1 \end{array} \right) \Longleftrightarrow U = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 10 \\ 0 & 0 & 1 & 3 & 7 \end{array} \right)$$

As before, the first four columns of U comprise the reduced row echelon form of A; that is,

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Inspecting the matrices, we see that $\text{rank}(A) = \text{rank}[A|\mathbf{b}] = 2 < 4$. By part 3 of Theorem 3.1, the system is consistent and has infinitely many solutions.

We close by pointing out that for any system $[A|\mathbf{b}] \in \mathbb{R}^{m\times(n+1)}$, the first n columns of the reduced echelon form of $[A|\mathbf{b}]$ always comprise the reduced echelon form of A. As above, we can use this to easily find $\text{rank}(A)$ and $\text{rank}[A|\mathbf{b}]$, without any additional work.