# The Inverse of a Matrix

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## 1 Definition of the Inverse

Inverses are defined only for square matrices. Thus, we start with an $n \times n$ (square) matrix $A$. We say that an $n \times n$ matrix $B$ is an inverse for $A$ if and only if $AB = BA = I$, where $I$ is the $n \times n$ identity matrix.

The reason that we want to consider inverses for matrices is that they enable us to easily obtain solutions to linear systems of equations. If we want to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}, \mathbf{b}$ are in $\mathbb{R}^n$ or $\mathbb{C}^n$ (that is, they are real or complex $n \times 1$ columns), and if $B$ is an inverse for $A$, then consider this chain of equations:

$$\begin{aligned} B(A\mathbf{x}) &= B\mathbf{b} \\ (\underbrace{BA}_{I})\mathbf{x} &= B\mathbf{b} \\ I\mathbf{x} &= B\mathbf{b} \\ \mathbf{x} &= B\mathbf{b} \end{aligned}$$

The point is that if we know an inverse $B$ for $A$, then the solution to $A\mathbf{x} = \mathbf{b}$ is just $\mathbf{x} = B\mathbf{b}$.

An example of a matrix that *has* an inverse is $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. It is easy to check that $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ is inverse to $A$. Simply observe that

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There are two important questions that we want to answer. We will start with the easier of the two. *If a matrix $A$ has an inverse $B$, can it have another inverse $C \neq B$?*

The answer is no. Let $B$ and $C$ be inverses for $A$. Then, $AB = I$ and $CA = I$. Thus, $C = CI = C(AB) = (CA)B = IB = B$.

The second question is this. *Do all nonzero $n \times n$ matrices have inverses?* The answer is again no. To see why, suppose that there is a column vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$. If such an $A$ had an inverse $B$, then $\mathbf{0} = B\mathbf{0} = B(A\mathbf{x}) = (BA)\mathbf{x} = I\mathbf{x} = \mathbf{x}$. But $\mathbf{x} \neq \mathbf{0}$, so we have a contradiction. For example, the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$ can't have an inverse because $\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

There are several conditions equivalent to $A\mathbf{x} = \mathbf{0}$ having a nontrivial (i.e., $\mathbf{x} \neq \mathbf{0}$) solution. The columns of $A$ being linearly dependent is one, and the rank of $A$ being less than $n$ is another. We discussed these in our previous notes on the rank of a matrix.

There is still another condition that will prove useful as well. Namely, $A$ is *not* row equivalent to the identity matrix $I$.

Let's turn this around and look at what happens if $A$ *is* row equivalent to $I$. To say that the $n \times n$ matrix $A$ is row equivalent to $I$ is to say that that the reduced echelon form of $A$ *is* $I$, since $I$ is in its reduced echelon form. The identity matrix $I$ has $n$ leading entries, so $\text{rank}(A) = n$. On the other hand, if $\text{rank}(A) = n$, then its reduced echelon form has $n$ leading entries, and so must be the identity matrix. We have thus shown this.

**Lemma 1.1** *An $n \times n$ matrix $A$ is row equivalent to the identity matrix $I$ if and only if $\text{rank}(A) = n$.*

Of course, this also means that $A$ is not row equivalent to $I$ if and only if $\text{rank}(A) < n$. We now put all the conditions that we found in one place, and add to the list another, which involves determinants, that we will discuss later.

**Proposition 1.2** *Let $A$ be an $n \times n$ matrix. If any of the following equivalent conditions hold, then $A$ does* <u>not</u> *have an inverse:*

1. *$A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*
2. *The columns of $A$ are linearly dependent.*
3. *The rank of $A$ is less than $n$.*
4. *$A$ is* not *row equivalent to the identity.*
5. $\det(A) = 0$.

## 2 Finding the Inverse

Suppose that the $n \times n$ matrix $A$ has an inverse $B$, so $AB = BA = I$. This means that *none* of the five conditions in Proposition 1.2 can hold. In particular, $A$ must be row equivalent to $I$.

We want to give a method for finding the inverse $B$. We can express $B$ in terms of its columns, $B = (\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n)$. Because matrix multiplication is done by multiplying a row times a column, we see that

$$AB = (A\mathbf{x}_1\ A\mathbf{x}_2\ \cdots\ A\mathbf{x}_n) = I = (\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_n).$$

Finding the columns of $B$ – and, hence, $B$ itself – amounts to solving the linear systems $A\mathbf{x}_1 = \mathbf{e}_1,\ A\mathbf{x}_2 = \mathbf{e}_2,\ \ldots,\ A\mathbf{x}_n = \mathbf{e}_n$. The key observation is that we can solve all of these equations at once, because the row operations required to row reduce $[A|\mathbf{b}]$ are the same regardless of what $\mathbf{b}$ is. Now, since $A \Leftrightarrow I$, there are row operations that take $A$ to $I$. If we apply these row operations to $[A\,|\,I]$, then we will produce $[I\,|\,B]$.

Let's look at some examples. Take $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, which was the example we used earlier in section 1. Here is how the algorithm works. First form the augmented matrix

$$[A\,|\,I] = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right).$$

Next, row reduce this matrix:

$$\begin{aligned} R_2 = R_2 - R_1:\ [A\,|\,I] &\Leftrightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \\ R_1 = R_1 - 2R_2:\ [A\,|\,I] &\Leftrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right). \end{aligned}$$

Thus, the inverse of $A$, which we now denote by $A^{-1}$, is $A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

Let's try one that we know does *not* have an inverse. Let $A = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$. Form the augmented matrix

$$[A\,|\,I] = \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right),$$

and then row reduce it:

$$R_2 = R_2 - R_1:\ [A\,|\,I] \Leftrightarrow \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right).$$

The zeroes in the bottom row mean that the two systems $A\mathbf{x}_1 = \mathbf{e}_1,\ A\mathbf{x}_2 = \mathbf{e}_2$ are inconsistent, so no inverse can exist.

Our algorithm will work – i.e., produce a matrix $B$ – under the assumption that $A \Leftrightarrow I$. The matrix $B$ satisfies $AB = I$. But to be an inverse, it must also satisfy $BA = I$. In the example that we looked at, we can just do the multiplication to check to see that the second equation holds. In fact, we could do this each time we used the algorithm, but that would be a nuisance.

It's nice to know that we don't have to trouble ourselves. The reason is very simple. Suppose that we list the row operations required to go from $[A|I]$ to $[I|B]$, and then apply them in *reverse order* to $[B|I]$. This set of row operations will take $I$ back to $A$ *and* $B$ back to $I$. Thus, we will get $[B|I] \Leftrightarrow [I|A]$. Going back through our analysis of systems of equations, columns, and so on, we see that this proves that our other equation, $BA = I$, holds. The final outcome of our analysis is that $A^{-1}$ exists if and only if $A \Leftrightarrow I$. Combining this with Lemma 1.1 and Proposition 1.2 gives the result below.

**Theorem 2.1** *Let $A$ be an $n \times n$ matrix with real or complex entries. The following statements are equivalent.*

1. *$A^{-1}$ exists.*
2. *The only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*
3. *The columns of $A$ are linearly independent.*
4. $\mathrm{rank}(A) = n$.
5. *$A$ is row equivalent to $I$.*
6. $\det(A) \neq 0$.

We close by mentioning that our analysis also shows that $(A^{-1})^{-1} = A$. We leave it to the reader to supply the details.