Final Examination

Instructions: *Show all work in your bluebook. Calculators that do linear algebra or calculus are* not *allowed.*

1. Let $B = \begin{pmatrix} 1 & -3 & 2 & -2 & 2 \\ -1 & 3 & -2 & 1 & -3 \\ 2 & -6 & 4 & -3 & 5 \end{pmatrix}$.

   (a) **(7 pts.)** Find bases for the null space, the row space, and the column space of B.

   (b) **(3 pts.)** What are the rank and nullity of B? What should they add up to? Do they?

2. Consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

   (a) **(10 pts.)** Find the eigenvalues and eigenvectors of A.

   (b) **(5 pts.)** Determine whether A is diagonalizable. If it is, give a diagonal matrix D and an invertible matrix S for which $D = S^{-1}AS$.

3. Let $L : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $L(p) = \frac{d}{dx}\left((1-x^2)\frac{dp}{dx}\right)$.

   (a) **(5 pts.)** Show that L is linear.

   (b) **(10 pts.)** Take $B = \{1, x, x^2, x^3\}$ as a basis for $\mathcal{P}_3$. Find the matrix of L relative to B. From this matrix, read off the eigenvalues of L.

4. **(10 pts.)** Find the matrix R of a rotation through 30° about the direction $\mathbf{a} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}^T$. Leave your final result as a product of matrices.

5. **(10 pts.)** Find the Fourier series for the function $f(x) = 1 - 2x$, where $-\pi < x \le \pi$.

Please turn over.

6. **(10 pts.)** Consider the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$ on continuous functions $C[-1,1]$. Use the Gram-Schmidt process to turn $\{1, x, x^2\}$ into an orthonormal set relative to this inner product.

7. **(15 pts.)** Use the method of Frobenius to solve order $n = 2$ Bessel's equation, $x^2y'' + xy' + (x^2 - 2^2)y = 0$.

8. Let V be an inner product space and let $L : V \to V$ be a linear operator. L is self adjoint if for any pair of vectors $\mathbf{v}, \mathbf{w} \in V$, we have $\langle L(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, L(\mathbf{w}) \rangle$.

   (a) **(5 pts.)** Show that if L is self adjoint, then the eigenvectors of L corresponding to distinct eigenvalues are orthogonal.

   (b) **(5 pts.)** Let $V = C[-1,1]$, $L(v) = \frac{d}{dx}\left((1 - x^2)\frac{dv}{dx}\right)$, $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$. Show that L is self adjoint. (Hint: integrate by parts twice.)

   (c) **(5 pts.)** Use your answers to problems 3, 8a, and 8b to explain why the Legendre polynomials are orthogonal relative to $\langle f, g \rangle$.