Extra Problems - Math 311-200

1. Consider the following operators, spaces, and inner products. In each case, show that the operator is self adjoint.

   (a) $L = \dfrac{d^2}{dx^2}$
   $V = \{f \in C^{(2)}[2,4] \,|\, f(2) = 0, f(4) = 0\}$
   $\langle f, g\rangle = \displaystyle\int_2^4 f(x)g(x)dx$

   (b) $L = \dfrac{d^2}{dx^2} - \dfrac{d}{dx}$
   $V = \{f \in C^{(2)}[0,\infty) \,|\, f(0) = 0, f \text{ is "nice" at } \infty\}$
   $\langle f, g\rangle = \displaystyle\int_0^\infty f(x)g(x)e^{-x}dx.$

   (c) $L = \dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d}{dr}\right)$
   $V = \{f \in C^{(2)}[0,1] \,|\, f \text{ is "nice" at } 0,\ f'(1) = 0\}$
   $\langle f, g\rangle = \displaystyle\int_0^1 f(r)g(r)r^2dr$

2. In section 9.6B, the text uses Gauss's theorem to establish Green's first identity,

   $$\int_R f\,\nabla^2 g\,dV - \int_R g\,\nabla^2 f\,dV = \int_{\partial R}\left(f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\right)d\sigma.$$

   Define the inner product

   $$\langle f, g\rangle = \int_R f(\mathbf{x})g(\mathbf{x})dV,$$

   and use this identity to show that $L = \nabla^2$ is self adjoint on $V = \{f \in C^{(2)}(R) \,|\, f(\mathbf{x}) = 0, \mathbf{x} \in \partial R\}$, which, in words, consists of all twice continuously differentiable functions on R that vanish on the boundary of R, ∂R. This space comes up in a heat flow problem for material confined to a region R and kept in a heat bath at $0°$.

3. Use the method of Frobenius in the following differential equations to find the indicial equation, the recurrence relation, and the the first few terms of the series solution.

   (a) $x^2y'' + xy' + (x^2 - 1)y = 0$.

   (b) $9x^2y'' + (9x^2 + 2)y = 0$.

   (c) $25x^2y'' + 25xy' + (x^4 - 1)y = 0$