Quiz 2 – Key

Instructions: Show all work in the space provided. No notes, calculators, cell phones, etc. are allowed.

1. Define the terms below.

   (a) **(5 pts.)** nested sequence of intervals – p. 46.

   (b) **(5 pts.)** Cauchy sequence – p. 49.

2. **(15 pts.)** Suppose that $x \geq 0 \in \mathbb{R}$, $x_n \geq 0$, and $x_n \to x$. Prove that $\sqrt{x_n} \to \sqrt{x}$.

   Solution. There are two cases to deal with, $x = 0$ and $x > 0$. For the $x = 0$ case we will apply the definition of limit directly. We have $\lim_{n\to\infty} x_n = 0$, so for $\epsilon^2 > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n| = x_n < \epsilon^2$. Next, note that $0 \leq a < b$ implies $0 \leq \sqrt{a} < \sqrt{b}$, so for $n \geq N$ we have that $\sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon$. By definition, $\lim_{n\to\infty} \sqrt{x_n} = 0$. For the $x > 0$ case we have:

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \frac{|\sqrt{x_n}^2 - \sqrt{x}^2|}{\sqrt{x_n} + \sqrt{x}} \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{|x_n - x|}{\sqrt{x}} \quad (\text{– since } \sqrt{x_n} \geq 0). \end{aligned}$$

   Since $x_n \to x$ is equivalent to $|x_n - x| \to 0$, the result follows from the squeeze theorem and the last inequality.

3. **(10 pts.)** Show that $x_n = \frac{(n+4)\sin(n^2+n)}{n+1}$, where $n \in \mathbb{N}$, has a convergent subsequence.

   Solution. We will first show that the sequence is bounded:

$$|x_n| = \frac{(n+4)|\sin(n^2+n)|}{n+1} \leq \frac{n+4}{n+1} < \frac{4n+4}{n+1} = 4.$$

   By the Bolzano-Weierstrass Theorem, it then has a convergent subsequence.

4. **(15 pts.) (Monotone Convergence of Sequences)** Prove this: If $\{x_n\}$ is an increasing sequence that is bounded above, then $\{x_n\}$ has a finite limit.

Proof. Let $s = \sup_{n\in\mathbb{N}} x_n$, which exists because the sequence is bounded. By the approximation property for suprema, for every $\epsilon > 0$ we have an x_N in the sequence for which $s - \epsilon < x_N \leq s$. Now, since the sequnece is increasing and has s as an upper bound, we have $s - \epsilon < x_N \leq x_n \leq s$ for all $n \geq N$. Manipulating this inequality gives us $|x_n - s| < \epsilon$ whenever $n \geq N$. From the definition of limit, $\lim_{n\to\infty} x_n = s$.

Comment. The proof actually gives us more than the statement of the theorem indicates. In fact, it shows that if a sequence is increasing and bounded above, then

$$\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n \,.$$