Quiz 4

Instructions: Show all work in the space provided. Add separate sheets of paper if necessary. Due date: Wednesday, June 29.

1. Define the terms below.

   (a) **(5 pts.)** uniform continuity on a nonempty subset $E \subseteq \mathbb{R}$ – p. 80.

   (b) **(5 pts.)** extension of a function – p. 82.

2. **(15 pts.)** Prove this: If $f : [0, \infty) \to \mathbb{R}$ is continuous and if the $\lim_{x\to\infty} f(x) = L$ exists and is finite, then f is uniformly continuous on $[0, \infty)$. (Hint: look at your notes from 6/27/05.)

   Proof. Let $\varepsilon > 0$. Since $f(x) \to L$ as $x \to \infty$, we can find $M > 0$ such that when $x > M$ we have $|f(x) - L| < \varepsilon/2$. With this M, we also note that f is continuous on the closed, bounded interval $[0, 2M]$, and is therefore uniformly continuous on $[0, 2M]$. Thus for ε there is a $\delta_1 > 0$ such that whenever two points $x, t \in [0, 2M]$ satisfy $|x - t| < \delta_1$, we have $|f(x) - f(t)| < \varepsilon$. Next, choose $\delta = \min\{M/2, \delta_1\}$. Any two points $x, t \in [0, \infty)$ are either both in $[0, 2M]$, in which case we have $|f(x) - f(t)| < \varepsilon$, or both are in $[M, \infty)$. In this latter case, we have $x > M$ and $t > M$, so

$$\begin{aligned} |f(x) - f(t)| &= |(f(x) - L) - (f(t) - L)| \\ &\leq |f(x) - L| + |f(t) - L| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

   Thus, f is uniformly continuous.

3. **(10 pts.)** Show that $x \sin(1/x)$ is uniformly continuous on $(0, 1)$.

   Solution. Let $f(x) = x \sin(1/x)$ when $x \in (0, 1)$. Note that the function $x \sin(1/x)$ is continuous as long as $x \neq= 0$. Also, we have that $|x \sin(1/x)| \leq |x|$, so the squeeze theorem implies that $\lim_{x\to 0} = 0$. Define $g(0) := 0$, $g(1) := 1 \cdot \sin(1/1) = \sin(1)$, and $g(x) = f(x)$ for $x \in (0, 1)$. By what we've just said, g is an extension of f that is continuous on $[0, 1]$. Hence, f is uniformly continuous on $(0, 1)$.

4. **(15 pts.)** Prove this: Suppose that $f : [a, b) \to \mathbb{R}$ is continuous and bounded on $[a, b)$. If in addition f is increasing — i.e., $x_1 < x_2$ implies $f(x_1) \le f(x_2)$ —, then f is uniformly continuous on $[a, b)$. (Hint: first prove that $f(b-) = \lim_{x \to b-} f(x)$ exists and is finite.)

   Proof. Since f is bounded on $[a, b)$, we can let $L = \sup_{x \in [a,b)} f(x)$. We now apply the approximation property for suprema. For every $\varepsilon > 0$ there is a point $y_0 = f(x_0)$ in the range of f such that $L - \varepsilon < y_0 = f(x_0) \le L$. Choose $\delta = b - x_0 > 0$. Whenever $0 < b - x < \delta$, we have $x_0 < x$, so $L - \epsilon < f(x_0) \le f(x) \le L$; equivalently, $|f(x) - L| < \varepsilon$. Thus, $f(x) \to L$ as $x \uparrow b$. That is, $f(b-) = L$. Define the function $g(x)$ to be $f(x)$ for $x \in [a, b)$, and let $g(b) = f(b-) = L$. This makes g a continuous extension of f to $[a, b]$. Thus, f is uniformly continuous on $[a, b)$. (There are other ways to get the extension, but this is the fastest I know of.)