Test 1 – Key

Instructions: Show all work in the space provided. No notes, calculators, cell phones, etc. are allowed.

1. Define the term or state the theorem.

   (a) **(5 pts.)** Cauchy sequence. – $\{x_n\}$ is a Cauchy sequence if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ and $m \geq N$, $n, m \in \mathbb{N}$, imply $|x_n - x_m| < \varepsilon$.

   (b) **(5 pts.)** Nested sequence of intervals. – A sequence of intervals $\{I_n\}$ is nested if and only if it satisfies $I_1 \supseteq I_2 \supseteq \cdots I_n \supseteq \cdots$.

   (c) **(5 pts.)** $\lim_{x\to\infty} f(x)$. – Let $f : I \to \mathbb{R}$, where $I = [a, \infty)$. We say $\lim_{x\to\infty} f(x) = L$ if and only if for every $\varepsilon > 0$ there an $M \in \mathbb{R}$ such that $x > M$, with $x \in I$, implies $|f(x) - L| < \varepsilon$.

   (d) **(5 pts.)** Intermediate Value Theorem. – Let $a, b \in \mathbb{R}$ with $a < b$ and let $I \supseteq$ be an interval. Suppose that $f : I \to \mathbb{R}$ is continuous on I. If $f(a) \neq f(b)$ and y_0 is between $f(a)$ and $f(b)$, then there is a number $c \in (a, b)$ such that $y_0 = f(c)$.

2. **(10 pts.)** Show that $(1+1/n)^n \geq 2$. (Hint: use the binomial theorem.)

   Solution. By the binomial theorem, we have

$$
\begin{aligned}
(1 + 1/n)^n &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(1/n)^k \\
&= 1 + n \cdot (1/n) + \text{nonnegative terms} \\
&\geq 2
\end{aligned}
$$

3. **(10 pts.)** Find $\displaystyle\lim_{x\to 1-} \frac{|x^2 + 2x - 3|}{x^2 - 1}$.

   Solution.

$$
\begin{aligned}
\lim_{x\to 1-} \frac{|x^2 + 2x - 3|}{x^2 - 1} &= \lim_{x\to 1-} \frac{|x - 1|}{x - 1} \frac{|x + 3|}{x + 1} \\
&= \lim_{x\to 1-} (-1)\frac{|x + 3|}{x + 1} \quad \text{(since } x - 1 < 0\text{)} \\
&= -\frac{1 + 3}{1 + 1} = -2 \text{ (algebraic limit theorems)}
\end{aligned}
$$

4. **(15 pts.)** Let $x_{n+1} = \sqrt{2 + x_n}$, $x_1 = 3$. Show that $\{x_n\}$ is decreasing and converges to a limit $x > 0$. Find x.

   Solution. First of all, we will show that $x_n \geq 0$ for all n. This is true for $n = 1$, since $x_1 = 3 > 0$. If $x_n \geq 0$, then $x_{n+1} \geq \sqrt{2 + x_n} \geq \sqrt{2} \geq 0$. Induction then gives us the result. Also, we will use induction to show that $x_{n+1} - x_n \leq 0$. For $n = 1$, this is true, since $\sqrt{5} - 3 < 0$. Suppose that it's true for n. Then, since $x_{n+1} - x_n \leq 0$, we have

$$\begin{aligned} x_{n+2} - x_{n+1} &= \sqrt{2 + x_{n+1}} - \sqrt{2 + x_n} \\ &= \frac{x_{n+1} - x_n}{\sqrt{2 + x_{n+1}} + \sqrt{2 + x_n}} \leq 0, \end{aligned}$$

   and so it's true for $n + 1$. By induction, $x_{n+1} - x_n \leq 0$ holds for all $n \in \mathbb{N}$, and x_n is thus decreasing. The monotone convergence and comparison theorems for sequences imply that x_n converges to $x \geq 0$. Taking limits in the original equation yields $x = \sqrt{x + 2}$. By squaring, we get $x^2 - x - 2 = 0$, so $x = -1$ or $x = 2$. But $x \geq 0$, so $x = 2$.

5. **(15 pts.)** Let I be an open interval, with $0 \in I$, and let $f : I \to \mathbb{R}$ be continuous at $x = 0$. Suppose that $f(0) > 2$. Show that there is $\delta > 0$ such that $f(x) > 2$ when $|x| < \delta$.

   Solution. The easiest way to do this is to apply the sign-preserving lemma to $g(x) = f(x) - 2$. Since f is continuous at $x = 0$, so is g. Also, $g(0) = f(0) - 2 > 0$. The lemma then implies that there are $\delta > 0$ and $\varepsilon > 0$ such that $g(x) = f(x) - 2 > \varepsilon$ for $|x| < \delta$. Hence, for $|x| < \delta$, $f(x) > 2 + \varepsilon > 2$.

6. **(15 pts.)** **(Approximation Property for Suprema)**. Prove this: If $E \subset \mathbb{R}$ has a supremum s, then for every $\varepsilon > 0$ there is an $a \in E$ such that $s - \varepsilon < a \leq s$.

   Proof. Suppose not. Then for some $\epsilon_0 > 0$ the interval $(s - \epsilon_0, s]$ contains no points from E. Since s is the supremum for E, there are no points of E in (s, ∞), either. It follows that all $a \in E$ are in $(-\infty, s - \epsilon_0]$. Hence, $s - \epsilon_0$ is an upper bound for E. However, $s - \epsilon_0 < s$. This is a contradiction, since every upper bound for E is greater than or equal to s, the supremum.

7. **(15 pts.) (Uniform Continuity Theorem)**. Prove this: Let $a < b$ be finite real numbers, and let $I = [a, b]$. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

   Proof. Suppose not. Then for some $\epsilon_0 > 0$ and every $\delta > 0$ there are points $x, t \in I$ such that $|x - t| < \delta$ and $|f(x) - f(t)| \geq \varepsilon_0$. Set $\delta = 1, 1/2, \ldots, 1/n, \ldots$. Then, there are corresponding points x_n and t_n, $|x_n - t_n| < 1/n$, $x_n, t_n \in I$ such that $|f(x_n) - f(t_n)| \geq \varepsilon_0$. These points satisfy $|x_n - t_n| < 1/n$. Since both $x_n, t_n \in I$, they are bounded. By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a subsequence x_{n_k} that converges to $x \in I$. For the corresponding subsequence t_{n_k} we have this:

$$\begin{aligned} |t_{n_k} - x| &= |t_{n_k} - x_{x_k} + x_{x_k} - x| \\ &\leq |t_{n_k} - x_{x_k}| + |x_{x_k} - x| \\ &< \frac{1}{n_k} + |x_{x_k} - x|. \end{aligned}$$

   Since $n_k \to \infty$ and $x_{n_k} \to x$ as $k \to \infty$, the squeeze theorem for sequences implies $t_{n_k} \to x$ as $k \to \infty$. The function f is continuous at x; thus, $\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(t_{n_k}) = f(x)$, and

$$\lim_{k\to\infty} |f(x_{n_k}) - f(t_{n_k})| = 0.$$

   However, $|f(x_{n_k}) - f(t_{n_k})| \geq \varepsilon_0 > 0$. The comparison theorem then implies $\lim_{k\to\infty} |f(x_{n_k}) - f(t_{n_k})| \geq \varepsilon_0 > 0$, or $0 > 0$. This is a contradiction.

   (The proof used here differs from the one given in the text, which I presented in class. Instead, it follows up on a suggestion made during our discussion of the theorem.)