Test II

Instructions: *Show all work in your bluebook. You may use a calculator for numerical computations. You may* not *use a graphing calcular or a calculator that can do symbolics.*

1. **(20 pts.)** Let $h(t) = \begin{cases} e^{-2t} & t \geq 0, \\ 0 & t < 0 \end{cases}$ and $f(t) = \begin{cases} 1 & 0 \leq t \leq \pi, \\ 0 & t < 0 \text{ or } t > \pi \end{cases}$.
   If H is the filter $H[f] = h * f$, find $H[f]$. Is H causal? Find the *system function*, $\sqrt{2\pi}\hat{h}$.

2. Let $\mathcal{F}_n$ be the DFT for n-periodic sequences (signals).
   (a) **(10 pts.)** Briefly describe the FFT. Why is it *fast*?
   (b) **(10 pts.)** Find $\mathcal{F}_4[a]$ if $a = (-2, 1, 0, 1)$. Note: for $n = 4$, $\overline{w} = -i$.
   (c) **(10 pts.)** State the (circular) convolution theorem for the DFT, and use it and part **2b** above to find the eigenvalues of the circulant matrix A whose first column is $a^T = [-2\ 1\ 0\ 1]^T$.

3. **(20 pts.)** Define *multiresolution analysis* (MRA). In the case of the Haar MRA, state or define the following items: the approximation spaces, V_j, the scaling function ϕ, the two-scale (scaling) relation, and the wavelet ψ. State what H, L are in Fig. 1.

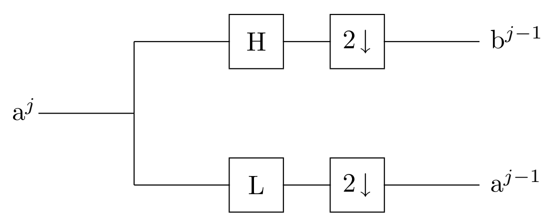

Figure 1: Haar decomposition diagram.

4. **(15 pts.)** Reconstruct $f \in V_3$, given these coefficients in its Haar wavelet decomposition:

$$a^1 = [3/2, -1] \quad b^1 = [-1, -3/2] \quad b^2 = [-3/2, -3/2, -1/2, -1/2].$$

   The first entry in each list corresponds to $k = 0$. Sketch f.

5. **(15 pts.)** Prove the Sampling Theorem: *Let f be band-limited. That is, suppose that $\hat{f}(\lambda)$ is piecewise smooth, continuous, and that $\hat{f}(\lambda) = 0$ for $|\lambda| > \Omega$, where Ω is some fixed, positive frequency. Then f has the following series expansion:*

$$f(t) = \sum_{j=-\infty}^{\infty} f(j\pi/\Omega) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}. \tag{1}$$

Fourier Transform Properties

1. $\hat{f}(\xi) = \mathcal{F}[f](\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx.$

2. $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi.$

3. $\mathcal{F}[x^n f(x)](\xi) = i^n \hat{f}^{(n)}(\xi).$

4. $\mathcal{F}[f^{(n)}(x)](\xi) = (i\xi)^n \hat{f}(\xi).$

5. $\mathcal{F}[f(x-a)](\xi) = e^{-i\xi a} \hat{f}(\xi).$

6. $\mathcal{F}[f(bx)](\xi) = \dfrac{1}{b} \hat{f}(\dfrac{\xi}{b}).$

7. $\mathcal{F}[f * g] = \sqrt{2\pi} \hat{f}(\xi) \hat{g}(\xi)$

8. $\mathrm{sinc}(x) := \dfrac{\sin(\pi x)}{\pi x} = \mathcal{F}^{-1}[\chi_\pi]$, if $\chi_\pi(\xi) = \begin{cases} 1/\sqrt{2\pi}, & -\pi \le \xi \le \pi \\ 0, & |\xi| > \pi \end{cases}$.

Integrals

1. $\int u dv = uv - \int v du$

2. $\int \dfrac{dt}{t} = \ln \big|t\big| + C$

3. $\int e^{at} dt = \frac{1}{a} e^{at} + C$

4. $\int t^n e^{at} dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} dt$

5. $\int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \cos(bt) + b \sin(bt)\right) + C$

6. $\int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \sin(bt) - b \cos(bt)\right) + C$

7. $\int t \sin(t) dt = \sin(t) - t \cos(t) + C$

8. $\int t \cos(t) dt = \cos(t) + t \sin(t) + C$

9. $\int \tan(at) dt = \dfrac{1}{a} \ln \big| \sec(at) \big| + C$

10. $\int \cot(at) dt = \dfrac{1}{a} \ln \big| \sin(at) \big| + C$

11. $\int \sec(at) dt = \dfrac{1}{a} \ln \big| \sec(at) + \tan(at) \big| + C$

12. $\int \csc(at) dt = \dfrac{1}{a} \ln \big| \csc(at) - \cot(at) \big| + C$

13. $\int \dfrac{dt}{t^2 + a^2} = \dfrac{1}{a} \arctan(t/a) + C$