Test 1

Instructions: *Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed.*

1. **(15 pts.)** Let $\langle f, g\rangle := \int_0^1 f(x)g(x)dx$. You are *given* that $S = \{1, \sqrt{3}(2x-1)\}$ is an orthonormal set. Find the straight line that best fits e^x in $L^2[0,1]$, in the least squares sense.

2. Consider the function $f(x) := x$ defined on $0 \le x \le \pi$.

   (a) **(15 pts.)** Find the Fourier *cosine* series (FCS) for f.

   (b) **(5 pts.)** Sketch three periods of the pointwise limit of the FCS for f.

   (c) **(5 pts.)** Define the term *uniform convergence*. Is the FCS uniformly convergent? Why or why not?

   (d) **(5 pts.)** Define the term *convergence in the mean*. Is the FCS convergent in the mean? Why or why not?

3. **(15 pts.)** Find the Fourier series for $h(x) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & -\pi \le x < 0 \end{cases}$. Use it to sum the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$.

4. Let $g(x) = e^{\frac{1}{4}x}$ on $0 < x < 2\pi$

   (a) **(15 pts.)** Show that the complex form of the Fourier series for g is
   $$g(x) = \sum_{n=-\infty}^{\infty} \frac{2(e^{\pi/2}-1)}{\pi(1-4in)} e^{inx}, \ 0 < x < 2\pi.$$

   (b) **(10 pts.)** Sum the series $\sum_{n=-\infty}^{\infty} \frac{1}{1+16n^2}$, using the Fourier series above and the complex form of Parseval's Theorem.

5. **(15 pts.)** Do *one* of the following.

   (a) Let $V_N = \text{span}\{1, \cos(x), \sin(x), \ldots, \cos(Nx), \sin(Nx)\}$. Prove that the $L^2[-\pi,\pi]$ projection of $f \in L^2$ is the partial sum, $S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$.

   (b) Suppose that f and $f' = \frac{df}{dx}$ are continuous 2π-periodic functions having the Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ and $f'(x) = a_0' + \sum_{n=1}^{\infty} a_n' \cos(nx) + b_n' \sin(nx)$. Then, show that $a_0' = 0$, and that $a_n' = nb_n$, $b_n' = -na_n$, for $n \ge 1$.

Integrals

1. $\int e^{at}dt = \frac{1}{a}e^{at} + C$

2. $\int t^n e^{at}dt = \frac{1}{a}t^n e^{at} - \frac{n}{a}\int t^{n-1}e^{at}dt$

3. $\int t\sin(t)dt = \sin(t) - t\cos(t) + C$

4. $\int t\cos(t)dt = \cos(t) + t\sin(t) + C$

5. $\int e^{at}\cos(bt)dt = \frac{e^{at}}{a^2+b^2}\left(a\cos(bt) + b\sin(bt)\right) + C$

6. $\int e^{at}\sin(bt)dt = \frac{e^{at}}{a^2+b^2}\left(a\sin(bt) - b\cos(bt)\right) + C$

7. $\int \cos(at)\cos(bt)dt = \frac{\sin((a+b)t)}{2(a+b)} + \frac{\sin((a-b)t)}{2(a-b)} + C,\ \ a \neq b$

8. $\int \sin(at)\sin(bt)dt = \frac{\sin((a+b)t)}{2(a+b)} - \frac{\sin((a-b)t)}{2(a-b)} + C,\ \ a \neq b$

9. $\int \sin(at)\cos(bt)dt = -\frac{\cos((a+b)t)}{2(a+b)} - \frac{\cos((a-b)t)}{2(a-b)} + C,\ \ a \neq b$