Notes on Daubechies' Wavelets by F. J. Narcowich

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The Daubechies Wavelets We want to find the c_k's (scaling coefficients) in the Daubechies' $N = 2$ case. In general, the two-scale relation has the form

$$\phi(x) = \sum_k c_k \phi(2x - k).$$

The Fourier transform of this equation is

$$\hat{\phi}(\xi) = P(e^{-i\xi/2})\hat{\phi}(\xi/2),$$

where $P(\cdot)$ is given by

$$P(z) = \frac{1}{2}\sum_k c_k z^k$$

One can also obtain the Fourier transform of the wavelet. Recall that the wavelet is given by the expansion

$$\psi(x) = \sum_k (-1)^k c_{1-k} \phi(2x - k). \tag{1}$$

Taking the Fourier transform of both sides yields

$$\hat{\psi}(\xi) = Q(e^{-i\xi/2})\hat{\phi}(\xi/2), \text{ where } Q(z) = \frac{1}{2}\sum_k (-1)^k c_{1-k} z^k = -zP(-z^{-1}).$$

Mallat's original thinking in defining an MRA was that the spaces and scaling functions were primary objects, and the scaling coefficients, the c_k's were derived from them. However, he did give a way to start with coefficients and obtain an MRA from them. To do that, there are three conditions that $P(z)$ must satisfy:

1. $|P(z)|^2 + |P(-z)|^2 \equiv 1$, $|z| = 1$.
2. $P(1) = 1$.
3. $|P(e^{-it})| > 0$ for $|t| \leq \pi/2$.

Note that #1, with $z = 1$, gives $|P(1)|^2 + |P(-1)|^2 = 1$. By #2, $P(1) = 1$, and so $1^2 + |P(-1)|^2 = 1$, from which it follows that

$$P(-1) = 0.$$

When there are only a finite number of non-zero c_k's, P is a polynomial. Since $z = -1$ is a root of P, we see that $P(z)$ has $(z+1)^N$, for some N, as a factor; that is,

$$P(z) = (z+1)^N \widetilde{P}(z), \quad \widetilde{P}(-1) \neq 0,$$

where $\widetilde{P}(z)$ is the product of the remaining factors of P after dividing out $z+1$ an appropriate number of times.

Let us return to the simplest case of a Daubechies wavelet, where there are four scaling coefficients and $P(z)$ is a cubic polynomial

$$P(z) = \frac{1}{2}\left(c_0 + c_1 z + c_2 z^2 + c_3 z^3\right). \tag{2}$$

that satisfies the three conditions listed above. The values N can have are 1, 2, or 3. It turns out that $N = 1$ gives the Haar case ($c_0 = c_1 = 1$, $c_2 = c_3 = 0$), and $N = 3$ doesn't work. If we take $N = 2$, then

$$P(z) = (z+1)^2(\alpha + \beta z),$$

where α and β are also assumed to be real. From #2, $1 = (1+1)^2(\alpha+\beta)$, so $\alpha + \beta = 1/4$. Hence, we see that P has the form

$$P(z) = (z+1)^2(1/4 - \beta + \beta z)$$

The question remaining is, does P satisfy #1 and #3? To begin, we will try to find a β for which #1 is satisfied. We do this simply by finding a value that works for $z = i$ ($|i| = 1$), and check to see if it works for all z with $|z| = 1$. We have

$$P(i) = (1+i)^2(1/4 - \beta + \beta i) = 2i(1/4 - \beta + \beta i) = -2\beta + (1/2 - 2\beta)i$$

Similarly, $P(-i) = -2\beta - (1/2 - 2\beta)i$. Consequently,

$$|P(i)|^2 + |P(-i)|^2 = 2(-2\beta)^2 + 2(1/2 - 2\beta)^2 = 16\beta^2 - 4\beta + 1/2$$

Since the left side is 1 by #1, we end up with $16\beta^2 - 4\beta + 1/2 = 1$ or $16\beta^2 - 4\beta - 1/2 = 0$. The roots of this equation are $\beta_\pm = \frac{1\pm\sqrt{3}}{8}$. It turns out that both values of β provide appropriate c_k's. In fact, the scaling functions they lead to are related to one another by a simple reflection of the x axis about the line $x = 3/2$. If we choose the "$-$", then

$$\begin{aligned} P(z) &= \frac{1}{8}(1+z)^2\Big((1+\sqrt{3}) + (1-\sqrt{3})z\Big) \\ &= \frac{1}{2}\Big(\underbrace{\frac{1+\sqrt{3}}{4}}_{c_0} + \underbrace{\frac{3+\sqrt{3}}{4}}_{c_1} z + \underbrace{\frac{3-\sqrt{3}}{4}}_{c_2} z^2 + \underbrace{\frac{1-\sqrt{3}}{4}}_{c_3} z^3\Big). \end{aligned}$$

These are the c_k's given in the text.

Showing that $P(z)$ satisfies #1 in our list requires some algebra, but is not really very hard. Verifying #3 is even easier. The only points at which $|P(z)| = 0$ are precisely the roots of P; namely, $z = -1$ (a double root) and $z = \frac{1+\sqrt{3}}{\sqrt{3}-1} \approx 3.7$. The root at $z = -1 = e^{i\pi}$ has angle $t = \pi > \pi/2$, so #3 holds in that case. The root at $z \approx 3.7$ has $|z| > 1$, so #3 holds there as well. Thus, for all $|t| \le \pi/2$, we have that $|P(e^{-it})| > 0$.

Moments and Quadrature Let $\rho : \mathbb{R} \to \mathbb{R}$. We define the k^{th} moment of ρ via the integral

$$m_k(\rho) = \int_{-\infty}^{\infty} x^k \rho(x)dx,$$

where we assume $x^k\rho(x) \in L^1(\mathbb{R})$. (The function ρ doesn't have to be positive.) It is easy to show that if p is a degree n polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ and if ρ has $n+1$ moments, $m_0(\rho), \ldots, m_n(\rho)$, then

$$\int_{-\infty}^{\infty} p(x)\rho(x)dx = \sum_{k=0}^{n} a_k m_k(\rho). \qquad (3)$$

Proposition 0.1 *Let $\delta > 0$. Suppose that* $\operatorname{supp}(h) \subseteq [0,\delta]$ *and that the first $n+1$ moments of ρ exist. If $f(x)$ is in $C^{(n)}[0,\delta]$, then*

$$\left| \int_0^{\delta} f(x)\rho(x)dx - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} m_k(\rho) \right| \le \frac{\|f^{(n)}\|_{L^\infty[0,\delta]}}{n!} \|x^n \rho(x)\|_{L^1[0,\delta]}.$$

The point here is that the proposition above shows that the quadrature formula

$$\int_0^\delta f(x)\rho(x)dx \doteq \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} m_k(\rho)$$

is accurate to within the error $\frac{\|f^{(n)}\|_{L^\infty[0,\delta]}}{n!}\|x^n\rho(x)\|_{L^1[0,\delta]}$.

We want to apply this to estimate the Daubechies wavelet coefficients, where we will use (1), but shifted to the right by 1. This gives us this formula for the wavelet:

$$\psi(x) = c_3\phi(2x) - c_2\phi(2x-1) + c_1\phi(2x-2) - c_0\phi(2x-3).$$

The support of ψ is $[0,3]$. Here is the result we want.

Proposition 0.2 *For the Daubechies wavelet above, $m_0(\psi) = m_1(\psi) = 0$. Moreover, the wavelet coefficient b_k^j for a function $f \in C^{(2)}$ then satisfies the bound*

$$|b_k^j| \leq \underbrace{\sqrt{3^5/20}}_{<4} \cdot 2^{-2j} \|f''\|_{L^\infty[2^{-j}k,\, 3\cdot 2^{-j}k]}.$$

Proposition 0.3 *For the Daubechies scaling function above, $m_0(\phi) = 1$ and $m_1(\phi) = 3 - \sqrt{3}$. Moreover, the scaling coefficient a_k^j for a function $f \in C^{(2)}$ then satisfies the bound*

$$|a_k^j - f(2^{-j}k) - (3-\sqrt{3})f'(2^{-j}k)2^{-j}| \leq \underbrace{\sqrt{3^5/20}}_{<4} \cdot 2^{-2j} \|f''\|_{L^\infty[2^{-j}k,\, 3\cdot 2^{-j}k]}.$$

We close by remarking that the "wavelet crime" of approximating a_k^j with $f(2^{-j}k)$ results in an error of order 2^{-j} if f is $C^{(1)}$.