The Rank of a Matrix

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1 Rank and Solutions to Linear Systems

The *rank* of a matrix A is the *number of leading entries* in a row reduced form R for A. This also equals the *number of nonrzero rows* in R. For any system with A as a coefficient matrix, $\mathrm{rank}[A]$ is the *number of leading variables*. Now, two systems of equations are equivalent if they have exactly the same solution set. When we discussed the row-reduction algorithm, we also mentioned that row-equivalent augmented matrices correspond to equivalent systems:

Theorem 1.1 *If $[A|b]$ and $[A'|b']$ are augmented matrices for two linear systems of equations, and if $[A|b]$ and $[A'|b']$ are row equivalent, then the corresponding linear systems are equivalent.*

By examining the possible row-reduced matrices corresponding to the augmented matrix, one can use Theorem 1.1 to obtain the following result, which we state without proof.

Theorem 1.2 *Consider the system $A\mathbf{x} = \mathbf{b}$, with coefficient matrix A and augmented matrix $[A|b]$. As above, the sizes of $\mathbf{b}$, A, and $[A|\mathbf{b}]$ are $m \times 1$, $m \times n$, and $m \times (n + 1)$, respectively; in addition, the number of unknowns is n. Below, we summarize the possibilities for solving the system.*

i. $A\mathbf{x} = \mathbf{b}$ *is inconsistent (i.e., no solution exists) if and only if* $\mathrm{rank}[A] < \mathrm{rank}[A|b]$.

ii. $A\mathbf{x} = \mathbf{b}$ *has a unique solution if and only if* $\mathrm{rank}[A] = \mathrm{rank}[A|b] = n$.

iii. $A\mathbf{x} = \mathbf{b}$ *has infinitely many solutions if and only if* $\mathrm{rank}[A] = \mathrm{rank}[A|b] < n$.

To illustrate this theorem, let's look at the simple systems below.

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + x_2 &= -2 \end{aligned} \qquad \begin{aligned} 3x_1 + 2x_2 &= 3 \\ -6x_1 - 4x_2 &= 0 \end{aligned} \qquad \begin{aligned} 3x_1 + 2x_2 &= 3 \\ -6x_1 - 4x_2 &= -6 \end{aligned}$$

The augmented matrices for these systems are, respectively,

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & -2 \end{array}\right) \qquad \left(\begin{array}{cc|c} 3 & 2 & 3 \\ -6 & -4 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|c} 3 & 2 & 3 \\ -6 & -4 & -6 \end{array}\right).$$

Applying the row-reduction algorithm yields the row-reduced form of each of these augmented matrices. The results are, again respectively,

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 1 \\ 0 & 0 & 0 \end{array}\right).$$

From each of these row-reduced versions of the augmented matrices, one can read off the rank of the coefficient matrix as well as the rank of the augmented matrix. Applying Theorem 1.2 to each of these tells us the number of solutions to expect for each of the corresponding systems. We summarize our findings in the table below.

System	rank[A]	rank[$A\|b$]	n	# of solutions
First	2	2	2	1
Second	1	2	2	0 *(inconsistent)*
Third	1	1	2	∞

Homogeneous systems. A *homogeneous system* is one in which the vector $\mathbf{b} = \mathbf{0}$. By simply plugging $\mathbf{x} = \mathbf{0}$ into the equation $A\mathbf{x} = \mathbf{0}$, we see that every homogeneous system has at least *one* solution, the *trivial* solution $\mathbf{x} = \mathbf{0}$. Are there any others? Theorem 1.2 provides the answer.

Corollary 1.3 *Let A be an $m \times n$ matrix. A homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ will have a unique solution, the trivial solution $\mathbf{x} = \mathbf{0}$, if and only if* rank$[A] = n$. *In all other cases, it will have infinitely many solutions. As a consequence, if $n > m$—i.e., if the number of unknowns is larger than the number of equations—, then the system will have infinitely many solutions.*

Proof: Since $\mathbf{x} = \mathbf{0}$ is always a solution, case (i) of Theorem 1.2 is eliminated as a possibility. Therefore, we must always have rank$[A] =$ rank$[A|0] \le n$. By Theorem 1.2, case (ii), equality will hold if and only if $\mathbf{x} = \mathbf{0}$ is the *only* solution. When it does not hold, we are always in case (iii) of Theorem 1.2; there are thus infinitely many solutions for the system. If $n > m$, then we need only note that rank$[A] \le m < n$ to see that the system has to have infinitely many solutions. $\square$

2 Linear Independence and Dependence

A set of k vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is *linearly independent* (LI) if the equation

$$\sum_{j=1}^{k} c_j \mathbf{u}_j = \mathbf{0},$$

where the c_j's are scalars, has *only* $c_1 = c_2 = \cdots = c_k = 0$ as a solution. Otherwise, the vectors are *linearly dependent* (LD). Let's assume the vectors are all $m \times 1$ column vectors. If they are rows, just transpose them. Now, use the *basic matrix trick* (BMT) to put the equation in matrix form:

$$A\mathbf{c} = \sum_{j=1}^{k} c_j \mathbf{u}_j = \mathbf{0}, \text{ where } A = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots \mathbf{u}_k] \text{ and } \mathbf{c} = (c_1\ c_2\ \cdots\ c_k)^T.$$

The question of whether the vectors are LI or LD is now a question of whether the homogeneous system $A\mathbf{c} = \mathbf{0}$ has a nontrivial solution. Combining this observation with Corollary 1.3 gives us a handy way to check for LI/LD by finding the rank of a matrix.

Corollary 2.1 *A set of k column vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is* linearly independent *if and only if the associated matrix $A = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots \mathbf{u}_k]$ has* $\operatorname{rank}[A] = k$.

Example 2.2 *Determine whether the vectors below are linearly independent or linearly dependent.*

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{u_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \ \mathbf{u_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Solution. Form the associated matrix A and perform row reduction on it:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank of A is $2 < k = 3$, so the vectors are LD. The row-reduced form of A gives us additional information; namely, we can read off the c_j's for which $\sum_{j=1}^{k} c_j \mathbf{u}_j = \mathbf{0}$. We have $c_1 = \frac{2}{3}t$, $c_2 = -\frac{1}{3}t$, and $c_3 = t$. As usual, t is a parameter.