

Notes on Row Reduction

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1 The Row-Reduction Algorithm

The row-reduced form of a matrix contains a great deal of information, both about the matrix itself and about systems of equations that may be associated with it. To talk about row reduction, we need to define several terms and introduce some notation.

1. *Notation for row operations.* Row reduction is made easier by having notation for various row operations. This is the notation that we will use here.
 - (a) *Elementary multiplication.* $R = cR$ means multiply the current row R by $c \neq 0$ and make the result the new row R .
 - (b) *Elementary modification.* $R = R + cR'$ means multiply R' by c , then add cR' to R , and make the result the new row R .
 - (c) *Row rearrangement.* $R \leftrightarrow R'$ means interchange rows R and R' . (This can be derived from the other two.)
2. *Row equivalence.* A matrix A is *row equivalent* to a matrix B if A can be transformed into B using a finite number of elementary row operations. Since such operations are reversible, B is also row-equivalent to A , and we simply say that A and B are row equivalent; we write $A \Leftrightarrow B$.
3. *Leading entry.* The *leading entry* in a row is the first non-zero entry in a row. The leading entries in each row of M are underlined.

$$M = \begin{pmatrix} 0 & \underline{1} & 3 & 2 \\ \underline{2} & 4 & 0 & -1 \\ 0 & 0 & \underline{6} & 5 \end{pmatrix}$$

4. *Row-reduced form of a matrix.* A matrix is in *row-reduced form* if these hold:

- (a) In any non-zero row, the leading entry is 1.
- (b) Any column containing a leading entry for some row has zeros in all its other entries.

The matrix M above is *not* in row-reduced form. Here are two examples of matrices that *are* in row-reduced form.

$$\begin{pmatrix} 0 & \underline{1} & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & -4 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \underline{1} & 0 & 4 \\ 0 & 0 & \underline{1} & 3 \\ \underline{1} & 0 & 0 & 7 \end{pmatrix}.$$

Although we will not prove it here, every matrix is row equivalent to a matrix that is in row-reduced form. We call the process of finding the row-reduced matrix equivalent to a given matrix *row reduction*. Below is an algorithm for row-reducing a matrix. There are others as well.

The Algorithm

1. Starting on the left, find the first column that contains a leading entry. If there are several leading entries in this column, choose a convenient one—for example, an entry that is 1.
2. Divide the row containing the leading entry by that leading entry.
3. Use this row and elementary modification to zero-out the other entries in the column that you are working with.
4. Find the next column to the right that contains a leading entry. Repeat the steps above. Stop when there are no more columns with leading entries.

Reduced Echelon Form Often it's desirable to put the matrix in *reduced echelon form*. After the matrix is in row reduced form, use row rearrangement to put the zero rows at the bottom of the matrix and to put the other rows in a staircase arrangement. Here are the reduced echelon forms of two of the matrices from above.

$$\begin{pmatrix} 0 & \underline{1} & 0 & 0 & 3 \\ 0 & 0 & \underline{1} & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \underline{1} & 0 & 0 & 7 \\ 0 & \underline{1} & 0 & 4 \\ 0 & 0 & \underline{1} & 3 \end{pmatrix}.$$

Rank of a matrix. Before we give an example, we want to mention a very important quantity that one can obtain from the row-reduced form of a matrix; namely, the *rank* of a matrix. The *rank* of a matrix A , which we denote by $\text{rank}[A]$, is defined to be the number of leading entries in the row reduced form of A ; this is equal to the number of non-zero rows in the row-reduced form of A . One more thing: If, after row-reducing a matrix, a column contains a leading entry, then that column (which is a column in the original matrix) is called a *leading column*. The rank of A is obviously also equal to the number of leading columns in A .

An Example. Here is an example of how the algorithm works. Consider the matrix A shown below. Leading entries are underlined.

$$\text{(Starting matrix) } A = \begin{pmatrix} \underline{1} & 1 & 1 \\ \underline{2} & 4 & 6 \\ \underline{1} & -1 & -3 \end{pmatrix}.$$

We will choose the first row and use elementary modification to zero out the other entries in the first column.

$$R_2 = R_2 - 2R_1, \quad R_3 = R_3 - R_1 : \quad A \Leftrightarrow \begin{pmatrix} \underline{1} & 1 & 1 \\ 0 & \underline{2} & 4 \\ 0 & \underline{-2} & -4 \end{pmatrix}$$

We now move to the second column, where we have a choice of two leading entries. We will choose the one in row 2.

$$R_2 = \frac{1}{2}R_2 : \quad A \Leftrightarrow \begin{pmatrix} \underline{1} & 1 & 1 \\ 0 & \underline{1} & 2 \\ 0 & \underline{-2} & -4 \end{pmatrix}$$

Next, we will zero out the other entries in column 2.

$$R_1 = R_1 - R_2, \quad R_3 = R_3 + 2R_2 : \quad A \Leftrightarrow \begin{pmatrix} \underline{1} & 0 & -1 \\ 0 & \underline{1} & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

There are no leading entries in column 3, so this completes the algorithm. The last matrix on the right above is the reduced echelon form of A . From this matrix we see that $\text{rank}[A] = 2$ and that the leading columns of A are

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}.$$

2 Solving Systems of Equations

We now want to look at how to solve a linear system of equations using row reduction to implement the method of elimination. We begin with the system below.

$$\begin{aligned}x_1 - 3x_2 - x_3 - 3x_4 &= 0 \\ -x_1 + 3x_2 + 2x_3 + 4x_4 &= 1 \\ 2x_1 - 6x_2 + 4x_3 &= 6\end{aligned}$$

The coefficient matrix A and “right-hand side” b for this system are

$$A = \begin{pmatrix} 1 & -3 & -1 & -3 \\ -1 & 3 & 2 & 4 \\ 2 & -6 & 4 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix},$$

and the augmented matrix $[A|b]$ is given by

$$[A|b] = \left(\begin{array}{cccc|c} 1 & -3 & -1 & -3 & 0 \\ -1 & 3 & 2 & 4 & 1 \\ 2 & -6 & 4 & 0 & 6 \end{array} \right).$$

To solve the original system, we row reduce the augmented matrix:

$$R_2 = R_2 + R_1, \quad R_3 = R_3 - 2R_1 : \quad [A|b] \Leftrightarrow \left(\begin{array}{cccc|c} 1 & -3 & -1 & -3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 & 6 \end{array} \right)$$

$$R_1 = R_1 + R_2, \quad R_3 = R_3 - 6R_2 : \quad [A|b] \Leftrightarrow \left(\begin{array}{cccc|c} \underline{1} & -3 & 0 & -2 & \underline{1} \\ 0 & 0 & \underline{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The final matrix on the right above is actually the reduced echelon form of $[A|b]$. The underlined matrix elements are the leading entries. Identifying them tells us that the *leading variables* are x_1 and x_3 , the *nonleading variables* are x_2 and x_4 , and the rank of the augmented matrix is 2. The equivalent system reduces to two equations, $x_1 - 3x_2 - 2x_4 = 1$ and $x_3 + x_4 = 1$. Assigning parameters to the *nonleading* variables and solving for the *leading* variables, we end up with the solution in *parametric form*:

$$\begin{cases} x_1 = 1 + 3t_1 + 2t_2 \\ x_2 = t_1 \\ x_3 = 1 - t_2 \\ x_4 = t_2 \end{cases} \iff \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$