The Sampling Theorem

Francis J. Narcowich
Department of Mathematics
Texas A&M University

April 13, 1999

A function f is said to be frequency band-limited if its Fourier transform satisfies

$$\hat{f}(\omega) = 0$$

for $|\omega| > \Omega$, where Ω is some fixed, positive frequency. One therefore can write f as the inverse FT of $\hat{f}$ this way:

$$f(t) = \int_{-\Omega}^{\Omega} \hat{f}(\omega))e^{i\omega t} d\omega. \tag{1}$$

Band-limited functions are interesting because they occur in certain applications, at least approximately. For example, our ear can only hear a band-limited version of speech. This fact is made use of in processing certain long-distance telephone calls.

The largest oscillations involved in a band-limited function have (circular) frequencies restricted to the interval $[-\Omega, \Omega]$. Thus, on a time-interval $[t, t+\Delta t]$, where $\Delta t \ll 1/\Omega$, one does not expect the function to change very much. In fact, one can see this from equation (1). If we take $\Delta t \ll 1/\Omega$, then

$$f(t+\Delta t) = \int_{-\Omega}^{\Omega} \hat{f}(\omega))e^{i\omega t} e^{i\omega\Delta t} d\omega. \tag{2}$$

Since $\Delta t \ll 1\Omega$, and since $|\omega| \leq \Omega$, we have that the term

$$e^{i\omega\Delta t} \approx 1 + i\omega\Delta t,$$

with the second term contibuting very little to the exponential. Indeed, if we subtract equation (1) from (2), and take absolute values, we get

$$|f(t+\Delta t) - f(t)| \leq \int_{-\Omega}^{\Omega} |\hat{f}(\omega)||e^{i\omega\Delta t} - 1|d\omega \leq C|\Omega\Delta t| \tag{3}$$

This suggests that if we sample the function at times t_j such that $t_{j+1} - t_j \approx 1/\Omega$, we would have a good idea of what the function looks like, or at least have sufficient data to approximately reconstruct it. The following theorem tells us that in fact if we sample properly, we can exactly reconstuct the the band-limited signal.

Theorem 1 (Sampling Theorem) *Suppose that $\hat{f}(\omega)$ is piecewise smooth, continuous, and that it vanishes for $|\omega| > \Omega$, where Ω is some fixed, positive frequency. Then $f = \mathcal{F}^{-1}[\hat{f}]$ is completely determined by its values at the points $t_j = j\pi/\Omega$, $j = 0, \pm 1, \pm 2, \ldots$. In fact, f has the following series expansion:*

$$f(t) = \sum_{j=-\infty}^{\infty} f(j\pi/\Omega) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi} \tag{4}$$

The frequency $\nu := \Omega/\pi$ is called the Nyquist frequency*.*

Proof: Expand $\hat{f}(\omega)$ in a Fourier series with period 2Ω,

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{i\pi k\omega/\Omega}, \quad c_k = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{-i\pi k\omega/\Omega} d\omega$$

Note that from (1) we have

$$c_k = \frac{1}{2\Omega} f(-k\pi/\Omega).$$

If we use this expression for c_k in the Fourier series for $\hat{f}$ above, and if at the same time we change the summation index from k to $j = -k$, we obtain

$$\hat{f}(\omega) = \sum_{j=-\infty}^{\infty} \frac{1}{2\Omega} f(j\pi/\Omega) e^{-i\pi j\omega/\Omega} \tag{5}$$

Because the $\hat{f}$ is a continuous, piecwise smooth function, the series (5) converges uniformly. We may therefore replace $\hat{f}$ in (1) with the series in (5), and then interchange the sum and integral that result. Doing so gives us

$$f(t) = \sum_{j=-\infty}^{\infty} \frac{1}{2\Omega} f(j\pi/\Omega) \int_{-\Omega}^{\Omega} e^{-i\pi j\omega/\Omega + i\omega t} d\omega \tag{6}$$

The integral in (6) is straightforward to do. Evaluating it yields

$$\int_{-\Omega}^{\Omega} e^{-i\pi j\omega/\Omega + i\omega t} d\omega = 2\,\frac{\Omega \sin(\,t\,\Omega - j\,\pi\,)}{t\,\Omega - j\,\pi}$$

Replace the integral in (6) with the expression above, and then simplify. This yields (4), which completes the proof. $\square$

We close by pointing out that the Nyquist frequency $\nu = \Omega/\pi$ represents the minimum sampling rate needed to reconstruct f. Sampling at a lower rate results in *aliasing*, where the function obtained will *not* be the one that was sampled. Some of the higher frequncies will be missing.