APPLIED ANALYSIS QUALIFYING EXAMINATION
JANUARY 2009

Hand in all the problems that you attempt. Your grade will be based on your best 7 answers.

Policy on misprints. *The qualifying examination committee tries to proofread the examinations as carefully as possible. Nevertheless, there may be a few misprints. If you are convinced that a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.*

Q1. (a) Define the notions of "test functions" and "distributions".

(b) Let T be a distribution. Then show (with convergence in the sense of distributions) that

$$T'(x) = \lim_{h\to 0} \frac{T(x+h) - T(x)}{h}$$

(c) Solve (in the sense of distributions) the equation $x\frac{du}{dx} = u$. (Hint: Consider the substitution u = xv).

Q2. Consider the equation $Lu = f,\ \ \lambda_1(u) = 0,\ \ \lambda_2(u) = 0$, where L is a second order linear differential operator. A Green's function $g(x, y)$ for L must satisfy $\lambda_1(g(x, y)) = 0,\ \ \lambda_2(g(x, y)) = 0$ where y is fixed and g is considered as a function of x.

(a) List the other properties $g(x, y)$ must satisfy.

(b) Consider the equation $u''(x) = f(x),\ \ u(0) = 0,\ \ \int_0^1 u(t)dt = 0$. Find the Green's function for this equation. (Hint: the Green's function has the form $u_1(\cdot)u_2(\cdot)$ where u_1 is a solution to $u''(x) = 0,\ \ u(0) = 0$ while u_2 is a solution to $u''(x) = 0$.

(c) Write down a solution to $u''(x) = f(x),\ \ u(0) = 0,\ \ \int_0^1 u(t)dt = 0$.

Q3. (a) State the Courant Minimax Principle.

(b) Prove an inequality relating the eigenvalues of a symmetric matrix before and after one of its diagonal elements is increased.

(c) Use this inequality and the minimax principle to show that the smallest eigenvalue of

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 3 \end{pmatrix}$$

is less than zero.

Q4. (a) Let K be a compact, self-adjoint operator on a Hilbert space H and suppose $(I - \lambda K)$ is bounded below, i.e., $\inf_{||u||=1} ||(I - \lambda K)u|| > 0$. Explain why $(I - \lambda K)u = f$ can always be solved whenever $f \in H$.

(b) With the same set-up as in (a), explain how to solve $(I - \lambda K)u = f$ explicitly in terms of the eigenfunctions of K.

Q5. (a) Prove the following theorem: If $\{P_n\}$ is a sequence of projections with the property that $||P_n u - u|| \to 0$ as $n \to \infty$ for every $u \in H$, and if $(I - \lambda K)^{-1}$ exists, then u_n, the solution of $(I - \lambda K)u_n = P_n f$, converges to the solution of $(I - \lambda K)u = f$ as $n \to \infty$.

(b) Apply this theorem to sketch a way to find an approximate solution of the integral equation

$$u(x) + \int_0^1 k(x, y)u(y)dy = f(x)$$

using piecewise linear finite elements. For simplicity assume $k(x, y)$ and $f(x)$ are continuous functions of their arguments and define $\phi_k(x)$ to be the piecewise linear continuous functions with $\phi_k(x_j) = \delta_{k,j}$ and linear on all the intervals $[x_j, x_{j+1}]$ where $x_j = j/n$. Also assume that $\{P_n\}$ are interpolating projections.

Q6. (a) Let f be a C^2 map from R^3 into R. State a necessary condition for a function $y \in C^1[a, b]$ to minimize $\int_a^b f(x, y(x), y'(x))dx$ subject to $y(a) = \alpha,\ y(b) = \beta$. What conditions are necessary when you minim-imize the same integral but with just an initial condition: $y(a) = \alpha$?

(b) Among all the functions $y \in C^1[a,b]$ that satisfy $y(a) = \alpha,\ y(b) = \beta$, find the one for which $\int_a^b u(t)^2 y'(t)^2 dt$ is a minimum. Here u is given as an element of $C[a,b]$.

Q7. Find two terms of the asymptotic expansion of

$$I(x) = \int_0^\infty t^x e^{-t} \ln t dt.$$

Q8. (a) Let X and Y be Banach spaces and let $\Omega \subseteq X$ be an open set. If $F : \Omega \to Y$ is continuous, define the Frechet derivative of F at $x_0 \in \Omega$.

(b) Define $F : C[0,1] \to C[0,1]$ by the equation $(F[x])(t) = x(t) + \int_0^1 [x(st)]^2 ds$. Compute $F'(x)$.

(c) Let X and Y be Banach spaces and $f : X \to Y$. Prove that if f is differentiable at x, then f is Lipschitz continuous at x. This means that $||f(y) - f(x)|| \leq \lambda ||x - y||$ for some λ and all y in a neighborhood of x.

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Q9. Find the spectral representation of the delta function for the operator

$$Lu = -u'', x \in [0, \infty), \;\; u'(0) = u(0)$$