Applied/Numerical Analysis Qualifying Exam

August 11, 2016

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
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Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = \phi''(x)$ for some $\phi \in \mathcal{D}$ if and only if $\int_{-\infty}^{\infty} \psi(x)dx = \int_{-\infty}^{\infty} x\psi(x)dx = 0$.
(d) Use 2(c) to solve, in the distributional sense, the differential equation $u'' = 0$.

Problem 2. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having u'' in $L^2[0,\infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0,\infty) \,|\, u'' \in L^2[0,\infty) \text{ and } u'(0) = 0\}.$$

(a) Find the Green's function $G(x, \xi; z)$ for $-G'' - zG = \delta(x - \xi)$, with $G_x(0, \xi; z) = 0$.
(b) Employ the spectral theorem (Stone's formula) to obtain the cosine transform formulas:

$$F(\mu) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\mu x)dx \text{ and } f(x) = \int_0^\infty F(\mu)\cos(\mu x)d\mu\,.$$

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) Consider $K \in \mathcal{C}(\mathcal{H})$. Show that if $\{\phi_n\}_{n=0}^\infty$ is an orthonormal set in $\mathcal{H}$, then $\lim_{n\to\infty} K\phi_n = 0$.
(b) Suppose that $K \in \mathcal{C}(\mathcal{H})$ is self adjoint.
    (i) Show that $\sigma(K)$ (the spectrum) consists only of eigenvalues, together with 0, and that the only limit point of $\sigma(K)$ is 0.
    (ii) Given that $\|K\| = \sup_{\|u\|=1} |\langle Ku, u\rangle|$, show that either $\|K\|$ or $-\|K\|$ (or possibly both) is an eigenvalue of K, and that the corresponding eigenspace is finite dimensional.

Problem 4. Suppose that $f(x)$ is 2π-periodic function in $C^{(m)}(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2π-periodic. Here $m > 0$ is a fixed integer. Let c_k denote the k^{th} (complex) Fourier coefficient for f, and let $c_k^{(j)}$ denote the k^{th} (complex) Fourier coefficient for $f^{(j)}$.

(a) Prove that $c_k^{(j)} = (ik)^j c_k,\ j = 0, \dots, m+1$. (Note: using term by term differentiation of the Fourier series *assumes* what you want to prove.)
(b) For $k \neq 0$, show that c_k satisfies the bound

$$|c_k| \le \frac{1}{2\pi|k|^{m+1}}\|f^{(m+1)}\|_{L^1[0,2\pi]}.$$

(c) Let $f_n(x) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the n^{th} partial sum of the Fourier series for f, $n \ge 1$. Show that

$$\|f - f_n\|_{L^2[0,2\pi]} \le C\frac{\|f^{(m+1)}\|_{L^1[0,2\pi]}}{n^{m+\frac{1}{2}}},$$

where C is independent of f and n.