Instructions: Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.

Problem 1. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.
(a) Consider $K \in \mathcal{C}(\mathcal{H})$. Show that if $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal set in $\mathcal{H}$, then $\lim_{n \to \infty} K \phi_n = 0$.
(b) Suppose that $K \in \mathcal{C}(\mathcal{H})$ is self adjoint. Let $\lambda \neq 0$ be an eigenvalue of $K$. Show that the corresponding eigenspace is finite dimensional.
(c) Given that $\|K\| = \sup_{\|u\|=1} |\langle Ku, u \rangle|$, show that either $\|K\|$ or $-\|K\|$ (or possibly both) is an eigenvalue of $K$.
(d) Briefly explain how (b) and (c) are used to develop the spectral theory of compact self adjoint operators. (Two sentences will suffice.)

Problem 2. Let $\mathcal{P}$ be the set of all polynomials.
(a) State and sketch a proof of the Weierstrass approximation theorem.
(b) Let $\mathcal{H} = L^2_w[0,1]$, where the inner product is $\langle f, g \rangle = \int_0^1 f(x)g(x)w(x)dx$ and where $w \in C[0,1]$, $w(x) \geq c > 0$ on $[0,1]$. Show that $\mathcal{P}$ is dense in $L^2_w[0,1]$. (You may use the density of $C[0,1]$ in $L^2[0,1]$.)
(c) Let $U := \{p_n\}_{n=0}^{\infty}$ be the orthonormal set of polynomials obtained from $\mathcal{P}$ via the Gram-Schmidt process. Show that $U$ is a complete orthonormal set in $L^2_w[0,1]$.

Problem 3. Suppose that $Tu(x) := \int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy$.
(a) Show that $T$ is a bounded operator on $L^2(\mathbb{R})$.
(b) You are given that the set $\phi_j = \chi_{[j,j+1]}$ is an orthonormal basis for $L^2(\mathbb{R})$. Show that $\|T\phi_j\| = \|T\phi_0\|$.
(c) Is $T$ compact? Prove your answer.

Problem 4. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having $u''$ in $L^2[0,\infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, $L$ has the domain $\mathcal{D}_L = \{u \in L^2[0,\infty) \mid u'' \in L^2[0,\infty) \text{ and } u'(0) = 0\}$.
(a) Find the Green’s function $G(x, y; \lambda)$ for $-G'' - \lambda G = \delta(x-y)$, with $G'(0, y; \lambda) = 0$ and $\lambda \in \mathbb{C} \setminus \{0, \infty\}$.
(b) Is $G$ a Hilbert-Schmidt kernel? Prove your answer.