**Combined Applied Analysis/Numerical Analysis Qualifier**
**Applied Analysis Part**
**August 6, 2024**

**Instructions:** *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

**Problem 1.** Recall that the DFT and inverse DFT are given by $\hat{y}_k = \sum_{j=0}^{n-1} y_j \bar{w}^{jk}$ and $y_j = \frac{1}{n}\sum_{j=0}^{n-1} \hat{y}_k w^{jk}$, where $w = e^{2\pi i/n}$.

(a) State and prove the Convolution Theorem for the DFT.

(b) Let $a$, $x$, $y$ be column vectors with entries $a_0, \ldots, a_{n-1}$, $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1}$. In addition, let $\alpha$, $\xi$ and $\eta$ be n-periodic sequences, the entries for one period, $k = 0, \ldots, n-1$, being those of $a$, $x$, and $y$, respectively. Consider the circulant matrix

$$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}.$$

Show that the matrix equation $Ax = y$ is equivalent to convolution $\eta = \alpha * \xi$.

(c) Use parts (a) and (b) above to show that the eigenvalues of $A$ are the entries in $\widehat{\alpha}$.

**Problem 2.** Let $\mathcal{H}$ be a (separable) Hilbert space and and let $S$ be a subset of $\mathcal{H}$.

(a) Define these: *$S$ is compact subset of $\mathcal{H}$* and *$S$ is a precompact subset of $\mathcal{H}$.*

(b) Consider the Hilbert space $\mathcal{H} = \ell^2$ and let

$$S = \{x = (x_1\ x_2\ x_3\ \ldots) \in \ell^2 : \sum_{n=1}^{\infty} n^2 |x_n|^2 < 1\}.$$

Show that $S$ is a precompact subset of $\ell^2$.

**Problem 3.** Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$.

(a) State and prove the Fredholm alternative.

(b) State the Closed Range Theorem.

(c) Let $\mathcal{H}$ be $L^2[0,1]$ and consider the kernel $k(x,y) = x^4 y^{12}$ and its associated operator $Ku(x) = \int_0^1 k(x,y)u(y)dy$. Show that $K$ is a Hilbert-Schmidt operator and that $\|K\|_{\text{op}} \le 1/15$.

(d) Consider the operator $L = I - \lambda K$. Find all values of $\lambda$ such that $Lu = f$ has a unique solution for all $f \in L^2[0,1]$. For these values find the resolvent $(I - \lambda K)^{-1}$.

**Problem 4.** Let $\mathcal{D}$ and $\mathcal{D}'$ be the spaces of test functions and of distributions, respectively.

(a) Show that $\psi \in \mathcal{D}$ has the form $\psi = (x\phi)'$, where $\phi$ is also in $\mathcal{D}$, if and only if $\int_{-\infty}^{\infty} \psi(x)dx = \int_0^{\infty} \psi(x)dx = 0$.

(b) Find all $u \in \mathcal{D}'$ for which $\frac{d}{dx}\big(xu\big) = u$.