Name: ____________________________

Instructions: Do any three problems. Show all work clearly. State the problem that you are skipping. No extra credit for doing all four.

Problem 1. Let \( \mathcal{P} \) be the set of all polynomials.

(a) State and sketch a proof of the Weierstrass Approximation Theorem.\(^1\)
(b) Use (a) to show that \( \mathcal{P} \) is dense in \( L^2[0,1] \). (You may use the the fact that \( C[0,1] \) is dense in \( L^2[0,1] \).)
(c) Let \( U := \{ p_n \}_{n=0}^{\infty} \) be the orthonormal set of polynomials obtained from \( \mathcal{P} \) via the Gram-Schmidt process. Show that \( U \) is a complete set in \( L^2[0,1] \).

Problem 2. Let \( \mathcal{D} \) be the set of compactly supported functions defined on \( \mathbb{R} \) and let \( \mathcal{D}' \) be the corresponding set of distributions.

(a) Define convergence in \( \mathcal{D} \) and \( \mathcal{D}' \).
(b) Show that \( \psi \in \mathcal{D} \) satisfies \( \psi = \phi'' \) for some \( \phi \in \mathcal{D} \) if and only if
\[
\int_{-\infty}^{\infty} \psi(x) dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x\psi(x) dx = 0.
\]
(c) Find all distributions \( T \in \mathcal{D}' \) such that \( T''(x) = \delta(x+1) - 2\delta(x) + \delta(x-1) \).

Problem 3. Let \( \mathcal{H} \) be a Hilbert space and let \( \mathcal{C}(\mathcal{H}) \) be the set of compact operators on \( \mathcal{H} \).

(a) State and prove the Fredholm Alternative.
(b) State the Closed Range Theorem.
(c) Let \( \mathcal{H} = L^2[0,1] \). Define the kernel \( k(x,y) := x^3y^2 \) and let \( Ku(x) = \int_0^1 k(x,y) u(y) dy \). Show that \( K \) is in \( \mathcal{C}(\mathcal{H}) \).
(d) Let \( L = I - \lambda K \), \( \lambda \in \mathbb{C} \), with \( K \) as defined in part (c) above. Find all \( \lambda \) for which \( Lu = f \) can be solved for all \( f \in L^2[0,1] \). For these values of \( \lambda \), find the resolvent \( (I - \lambda K)^{-1} \).

Problem 4. Consider the kernel \( k(x,y) = \sum_{n=0}^{\infty} (1 + n)^{-2} P_{n+1}(x) P_n(y) \), where the \( P_n \)’s are the orthogonal set of Legendre polynomials, relative to \( L^2[-1,1] \). They are normalized so that \( \int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1} \).

(a) Show that \( Ku(x) = \int_{-1}^{1} k(x,y) u(y) dy \) is a compact operator on \( L^2[-1,1] \).
(b) Determine the spectrum of \( K \).

\(^1\)You may use these identities involving the Bernstein polynomials. The last two identities start the sum at \( j = 0 \), rather than \( j = 1 \).
\[
1 = \sum_{j=0}^{n} \beta_j,n(x), \quad x = \sum_{j=0}^{n} \frac{j}{n} \beta_j,n(x) \quad \frac{1}{n} x + (1 - \frac{1}{n})x^2 = \sum_{j=0}^{n} \frac{j^2}{n^2} \beta_j,n(x).
\]