NUMERICAL ANALYSIS QUALIFIER

January, 2020

Problem 1. Consider the following two finite elements: (τ, Q_1, Σ) and $(\tau, \widetilde{Q}_1, \Sigma)$, where

$$\begin{aligned}
&\tau = [-1,1]^2 \\
&Q_1 = span\{1, x, y, xy\}, \\
&\widetilde{Q}_1 = span\{1, x, y, x^2 - y^2\} \\
&\Sigma = \{w(-1,0), w(1,0), w(0,-1), w(0,1)\}.
\end{aligned}$$

Obviously, Σ is the set of the values of a function $w(x, y)$ at the midpoints of the edges of τ.

(a) Show that the finite element (τ, Q_1, Σ) is not unisolvent.
(b) Show that the finite element $(\tau, \widetilde{Q}_1, \Sigma)$ is unisolvent.
(c) Show that the finite element spaces are in general not H^1-conforming.

Problem 2. Consider the boundary value problem

$$\begin{aligned}
&u^{(4)}(x) + q(x)u = f(x), \qquad\qquad 0 < x < 1, \\
&u(0) = 0, \ u(1) = 0, \\
&u''(0) = -\gamma, \ u'(1) + u''(1) = \beta,
\end{aligned} \tag{2.1}$$

where $f(x)$ is a given function on $(0, 1)$, β and γ are given constants and $q(x) \geq 0$.

(a) Give a weak formulation of this problem in an appropriate space V, characterize V, and prove that the corresponding bilinear form is coercive on V.

(b) Set up a finite dimensional space $V_h \subset V$ of piece-wise cubic functions over a uniform partition of $(0, 1)$. Introduce the Galerkin finite element method for the problem (2.1) for V_h. State an error estimate in V-norm assuming that $u(x) \in H^4(0, 1)$ (do NOT prove this).
(c) Assuming "full regularity" and using duality argument **prove** the following estimate for the error of the Galerkin solution u_h:

$$\|u - u_h\|_{L^2} \leq Ch^4 \|u^{(4)}\|_{L^2}. \tag{2.2}$$

Further prove the estimate $\|u' - u_h'\|_{L^2} \leq Ch^3 \|u^{(4)}\|_{L^2}$.

Problem 3. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, and let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω with element diameters uniformly equivalent to h. Let also $V_h \subset H_0^1(\Omega)$ be a piecewise linear Lagrange finite element space. You may assume the existence of an interpolation operator $I_h : H_0^1(\Omega) \to V_h$ satisfying

$$\|u - I_h u\|_{L_2(\Omega)} + h\|u - I_h u\|_{H^1(\Omega)} \leq Ch^2 |u|_{H^2(\Omega)}.$$

(a) Let $u(t) \in H_0^1(\Omega)$ $(0 \leq t \leq T)$, u_0, and f be sufficiently smooth such that

$$\int_\Omega u_t v \ dx + \int_\Omega \nabla u \cdot \nabla v \ dx = \int_\Omega f v \ dx, \ v \in H_0^1(\Omega), \ 0 < t \leq T,$$
$$u(x, 0) = u_0(x), \ x \in \Omega.$$

Write down the spatially semidiscrete (i.e., discretized in space but not in time) finite element formulation of this problem. Denote by u_h the solution to these finite element equations.
(b) For $0 < t \leq T$, let now $\tilde{u}_h(t)$ be the *elliptic* finite element approximation to $u(t)$. That is

$$\int_\Omega \nabla \tilde{u}_h(t) \cdot \nabla v_h \ dx = \int_\Omega \nabla u(t) \cdot \nabla v_h \ dx, \ v_h \in V_h.$$

Prove that

$$\int_\Omega (u_h - \tilde{u}_h)_t v_h \; dx + \int_\Omega \nabla(u_h - \tilde{u}_h) \cdot \nabla v_h \; dx = \int_\Omega (u - \tilde{u}_h)_t v_h \; dx, \; v_h \in V_h, \; 0 < t \le T.$$

(c) Next recall Gronwall's Lemma, which states that if σ and ρ are continuous real functions with $\sigma \ge 0$ and $c \ge 0$ is a constant, and if

$$\sigma(t) \le \rho(t) + c \int_0^t \sigma(s) \; ds, \; t \in [0, T],$$

then

$$\sigma(t) \le e^{ct} \rho(t), \; t \in [0, T].$$

Using this result, prove that

$$\|(u_h - \tilde{u}_h)(T)\|^2_{L_2(\Omega)} \le C(T) \left(\|(u_h - \tilde{u}_h)(0)\|^2_{L_2(\Omega)} + \int_0^T \|(u - \tilde{u}_h)_t(s)\|^2_{L_2(\Omega)} \; ds \right).$$

(d) For the final part you will need the following intermediate result. Given $v \in H^1_0(\Omega) \cap H^2(\Omega)$, let $v_h \in V_h$ satisfy

$$\int_\Omega \nabla v_h \cdot \nabla w_h dx = \int_\Omega \nabla v \cdot \nabla w_h dx, \quad \text{all } w_h \in V_h.$$

Then

$$\|v - v_h\|_{L_2(\Omega)} \le C h^2 |v|_{H^2(\Omega)}.$$

Assuming this result and additionally that $\|(u - u_h)(0)\|_{L_2(\Omega)} \le C h^2 |u(0)|_{H^2(\Omega)}$, prove that

$$\|(u - u_h)(T)\|_{L_2(\Omega)} \le C(T) h^2 \left(|u(0)|_{H^2(\Omega)} + \left(\int_0^T |u_t|^2_{H^2(\Omega)} \right)^{1/2} \right).$$