NUMERICAL ANALYSIS QUALIFIER

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Problem 1. Let T be the unit triangle in $\mathbb{R}^2$, with vertices $v_1 = (0,0)$, $v_2 = (1,0)$, and $v_3 = (0,1)$ and edges $e_1 = v_1v_2$, $e_2 = v_2v_3$ and $e_3 = v_3v_1$. Let z_i be the midpoint of the edge e_i. Let $TW_0 = \{(a - cy, b + cx) : a, b, c \in \mathbb{R}\}$ (so that members of TW_0 are vector functions over T), and $[\mathbb{P}_0]^2 \subsetneq TW_0 \subsetneq [\mathbb{P}_1]^2$). Finally, let $\sigma_i(\vec{u}) = \vec{u}(z_i) \cdot \vec{t_i}$, where $\vec{t_i}$ is the counterclockwise-pointing unit vector tangent to ∂T on e_i, and let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$.

(a) Show that (T, TW_0, Σ) is a finite element triple.
(b) Find a basis $\{\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3\}$ for TW_0 that is dual to Σ, that is, $\sigma_i(\vec{\varphi}_j) = \delta_{ij}$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.
(c) Let $(\Pi\vec{u})(x) = \sum_{i=1}^{3} \sigma_i(\vec{u})\vec{\varphi}_i(x)$, $x \in T$ and $\vec{u} \in [H^2(T)]^2$. Show that

$$\|\vec{u} - \Pi\vec{u}\|_{[L_2(T)]^2} \leq C(|\vec{u}|_{[H^1(T)]^2} + |\vec{u}|_{[H^2(T)]^2}), \quad \vec{u} \in [H^2(T)]^2.$$

Note: You may use standard analysis results such as trace, Sobolev, and Poincarè inequalities and the Bramble-Hilbert Lemma without proof, but specify precisely which results you are using.

Problem 2. Consider the following initial boundary value problem: find a solution $u(x,t)$ such that

$$\begin{cases} \dfrac{\partial}{\partial t}\big(u - \Delta u\big) - \mu \Delta u = f, & \text{for } x \in \Omega,\ 0 < t \leq T, \\ u(x,t) = 0, & \text{for } x \in \partial\Omega,\ 0 < t \leq T, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\partial\Omega$ its boundary, $\mu > 0$ a given constant, and $f(x,t)$ and $u_0(x)$ are given right hand side and initial data functions.

In the following let $V = H_0^1(\Omega)$ and let $V_h \subset V$ be a finite element approximation space with (nodal) basis $\varphi_i^h(x)$, $i = 0, \ldots, \mathcal{N}$. Let $t_0 = 0 < t_1 < \ldots < t_N = T$ be a partition of $[0,T]$ into N uniform subintervals with time step size $k = t_{n+1} - t_n$.

(a) For given $u^n \in V$ at time t_n find the *semi-discrete weak formulation* of the initial boundary value problem where the forward Euler method is used to compute a value $u^{n+1} \in V$ at time t_{n+1}.

(b) Introduce matrices $M_h \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ with $(M_h)_{ij} = \big(\varphi_i^h, \varphi_j^h\big)_{L^2(\Omega)}$, and $A_h \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ with $(A_h)_{ij} = \big(\nabla\varphi_i^h, \nabla\varphi_j^h\big)_{L^2(\Omega)^3}$. Verify that the *fully discrete* scheme of the initial boundary value problem can be written as follows: Given a coefficient vector $U^n \in \mathbb{R}^{\mathcal{N}}$ at time t_n compute $U^{n+1} \in \mathbb{R}^{\mathcal{N}}$ for time t_{n+1} as follows:

$$\big(M_h + A_h\big)\frac{U^{n+1} - U^n}{k} + \mu A_h U^n = M_h F^n,$$

where the coefficient vector $F^n \in \mathbb{R}^n$ is formed by setting

$$\big(M_h F^n\big)_i = \big(f(.,t_n), \varphi_i^h\big)_{L^2(\Omega)}, \quad i = 1, \ldots, \mathcal{N}.$$

(c) Now introduce an orthonormal basis of eigenvectors $\Psi^j \in \mathbb{R}^{\mathcal{N}}$ with eigenvalues $\lambda_j > 0$ of the following generalized eigenvalue problem:

$$A_h\Psi^j = \lambda_j M_h \Psi^j, \quad \text{and} \quad (\Psi^j)^T M_h \Psi^j = \delta_{ij}, \quad \text{for } i, j = 1, \ldots, \mathcal{N}.$$

Here, δ_{ij} denotes the Kronecker delta. Expand

$$U^n = \sum_{j=1}^{\mathcal{N}} c_j^n \Psi^j, \quad F^n = \sum_{j=1}^{\mathcal{N}} d_j^n \Psi^j. \quad \text{and set} \quad \delta_j = \frac{1 + (1 - k\mu)\lambda_j}{1 + \lambda_j}.$$

Find the Courant (CFL) condition for stability and prove that

$$|c_j^{n+1}| \;\leq\; \delta_j\,|c_j^n| + \frac{k}{1+\lambda_j}|d_j^n| \quad \text{for } j = 1, \dots, \mathcal{N}.$$

(d) Derive a stability estimate that relates $|c_j^{n+1}|$ to the initial coefficient c_j^0 and right hand side coefficients d_j^ν, $\nu = 0, \dots, n$.

Problem 3. Consider the interval $D := (0,1)$. Let $\mu \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}$, $\nu \in \mathbb{R}_{>0}$ and $f \in L^1(D)$ *(note carefully what regularity is assumed of f)*. Consider the equation

$$\mu u(x) + \beta \partial_x u(x) - \nu \partial_{xx} u(x) = f(x), \quad \text{for a.e. } x \in D, \tag{3.1}$$

$$u(0) = a, \quad u(1) = b. \tag{3.2}$$

Let I be a positive natural number. Let $h := \frac{1}{I+1}$. Let $\mathcal{T}_h$ be the uniform mesh composed of the cells $[x_i, x_{i+1}]$, with $x_i := ih$, for all i in $\{0 \dots I+1\}$. Let $P_1(\mathcal{T}_h)$ be the Lagrange finite element space composed of the scalar-valued functions that are continuous and piecewise linear on the mesh $\mathcal{T}_h$. We also denote $P_{1,0}(\mathcal{T}_h) := P_1(\mathcal{T}_h) \cap H_0^1(D)$.

(a) Let $u_{ab}(x) = a(1-x) + bx$ be the natural linear lifting of the boundary conditions. Let $u_0(x) := u(x) - u_{ab}(x)$ so that $u_0(0) = 0$ and $u_0(1) = 0$. Write the weak form of the problem for u_0 where the trial and test spaces are $H_0^1(D)$. Use the norm $\|v\|_{H^1(D)} := (\|v\|^2_{L^2(D)} + \|\partial_x v\|^2_{L^2(D)})^{\frac{1}{2}}$.

(b) Prove that the proposed weak form of the problem is well-posed (*Hint:* You may invoke the boundedness of the embedding $H^s(D) \subset L^\infty(D)$ when $s > \frac{1}{2}$. Recall also that $\min(\mu, \nu) > 0$.)

(c) Write the Galerkin formulation of the weak formulation proposed in part (a) in the space $P_{1,0}(\mathcal{T}_h) := P_1(\mathcal{T}_h) \cap H_0^1(D)$ and denote by $u_{h,0}$ the approximation of u_0.

(d) Denote $u_h := u_{h,0} + u_{ab}$. Prove that

$$\|u - u_h\|_{H^1(D)} \leq C \inf_{\chi \in P_1(\mathcal{T}_h)} \|u - \chi\|_{H^1(D)}.$$

Explain why we *cannot* immediately conclude using the usual arguments that

$$\|u - u_h\|_{H^1(D)} \leq C(u)h,$$

where $C(u)$ depends on u. (Think carefully about what must be true about u in order for these estimates to hold.)