Applied Analysis/Numerical Analysis Qualifying Exam

January 10, 2019

Numerical Analysis Part, 2 hours

Name____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Question I.

Consider the variational problem: find

$$u \in H^1(\Omega) \equiv \mathbb{V}, \quad \text{s.t. } a(u,v) = L(v) \text{ for all } v \in \mathbb{V} \equiv H^1(\Omega). \tag{1}$$

Here $\Omega = (0,1) \times (0,1)$, $\Gamma = \partial\Omega$ is its boundary,

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Gamma uv \,\mathrm{d}s, \quad \text{and} \quad L(v) = \int_\Gamma gv \,\mathrm{d}s, \tag{2}$$

where g is a given smooth function of Γ.

(a) Derive the strong form of problem (1).

(b) Let $\mathcal{T}_h$ be a shape-regular partitioning of Ω into triangles. Introduce the finite dimensional space $\mathbb{V}_h$ consisting of continuous piecewise linear polynomials over $\mathcal{T}_h$. Show that $\mathbb{V}_h \subset \mathbb{V}$.

(c) Consider the finite element approximation of (1): find

$$u_h \in \mathbb{V}_h, \quad \text{s.t.} \quad a(u_h, v) = L(v) \quad \text{for all} \quad v \in \mathbb{V}_h. \tag{3}$$

State (not prove) the optimal estimate for the error $\|u - u_h\|_{\mathbb{V}}$ assuming that the solution to (1) belongs to the Sobolev space $H^2(\Omega)$. Derive a bound for $\|u - u_h\|_{L^2(\Omega)}$ under the assumption of full regularity of the problem (1).

(d) Assume that in the evaluation of the boundary term $\int_\Gamma u_h v \mathrm{d}s$ you have applied the composite trapezoidal quadrature rule:

$$\int_\Gamma f \,\mathrm{d}s \approx \sum_{e\in\Sigma} \frac{|e|}{2}(f(e_1) + f(e_2)) := \sum_{e\in\Sigma} Q_e(f),$$

where Σ is the set of boundary edges and for $e \in \Sigma$, e_1, e_2 are the endpoints of e (order is irrelevant) and $|e|$ is the length of e. In this way you have generated the approximate bilinear from

$$a_h(u_h, v) = \int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}x + \sum_{e\in\Sigma} Q_e(u_h v).$$

State the FEM using this approximation (this is one of the cases of variational "crimes"). Show that

$$a_h(v_h, v_h) \geq c\|v_h\|^2_{\mathbb{V}}, \qquad \forall v_h \in \mathbb{V}_h,$$

where c is a constant only depending on Ω.
Hint: Recall that there exists a constant C only depending on Ω such that for all $v \in \mathbb{V}$

$$C\int_\Omega v^2 \leq \int_\Omega |\nabla v|^2 + \int_\Gamma v^2.$$

(e) Show that

$$|a(u_h, v) - a_h(u_h, v)| \leq Ch\|u_h\|_{\mathbb{V}}\|v\|_{\mathbb{V}} \quad \text{for } u_h, v \in \mathbb{V}_h,$$

where C is a constant only depending on Ω.

Question II.

Consider the following initial boundary value problem: find $u(\cdot,t) := u(t) \in \mathbb{V}$, with $\mathbb{V} := H_0^1(\Omega)$, s.t.

$$(\frac{d}{dt}u(t),\phi) + (\nabla u(t),\nabla\phi) = (f(t),\phi), \quad \forall\phi\in\mathbb{V}, \quad t>0, \quad u(x,0)=u_0(x), \quad x\in\Omega, \tag{4}$$

where $u_0 : \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ are given functions and $f(t) := f(\cdot,t)$.

Let $\mathbb{V}_h \subset \mathbb{V} := H_0^1(\Omega)$ consists of continuous piecewise linear functions over a partition $\mathcal{T}_h$ of Ω into triangles.

(a) Consider the semi-discrete (in space) Galerkin finite element approximation of (4): find $u_h(t) \in \mathbb{V}_h$ s.t.

$$(\frac{d}{dt}u_h(t),\phi) + (\nabla u_h(t),\nabla\phi) = (f(t),\phi), \quad \forall\phi\in\mathbb{V}_h, \quad t>0, \quad u_h(0)=R_h u_0, \tag{5}$$

where $R_h u_0 \in \mathbb{V}_h$ satisfies

$$(\nabla R_h u_0, \nabla\phi) = (\nabla u_0,\nabla\phi), \qquad \forall\phi\in\mathbb{V}_h.$$

Prove that the solution $u_h(t)$ satisfies the a priori estimate

$$\|u_h(t)\|^2 \le \|u_h(0)\|^2 + c_0\int_0^t \|f(s)\|^2 ds, \quad t>0, \tag{6}$$

where c_0 is the constant in the Poincaré inequality $\|v\|^2 \le c_0\|\nabla v\|^2$.

(b) Let $k>0$ and set $t_n = nk$ for $n = 0,1,....$ The implicit Euler scheme approximating the problem (5) is given by: Set $U^0 = R_h u(0) = u_h(0)$, find $U^n \in \mathbb{V}_h$ recursively such that for $n = 1,\ldots$ it satisfies

$$\left(\frac{U^n - U^{n-1}}{k},\phi\right) + (\nabla U^n,\nabla\phi) = (f(t_n),\phi), \ \forall\phi\in\mathbb{V}_h.$$

Prove an a priori estimate for this fully discrete method that is similar to estimate (6):

$$\|U^n\|^2 \le \|U^0\|^2 + c_0\sum_{j=1}^n k\|f(t_j)\|^2.$$

Derive an a priori estimate for the error $e = u_h(t_n) - U^n$.

Question III.

Let Q be the three dimensional cube

$$Q = \Big\{(x_1,x_2,x_3)\in\mathbb{R}^3 \;\big|\; 0\le x_i\le 1, \quad i=1,2,3\Big\},$$

and let $\mathcal{Q}_2$ be the space of polynomials of degree 2 **in each direction**. Consider the point value evaluation functionals defined for any $p \in \mathcal{Q}_2$

$$\sigma_{i,j,k}(p) = p(i/2, j/2, k/2)$$

for $i,j,k = 0,1,2$ Show that this choice of Q, $\mathcal{Q}_2$, and degrees of freedom $\{\sigma_{i,j,k}\}$ is unisolvent.
Hint: you can use without proof the following result:

> *Let p be a polynomial of degree $d \ge 1$ that vanishes on the hyperplane given by the relation $h(x) = 0$. Then $p(x) = h(x)q(x)$, where q is a polynomial of degree $d-1$.*