ON ASSOUAD'S EMBEDDING TECHNIQUE

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Abstract. We survey the standard proof of a theorem of Assouad stating that every snowflaked version of a doubling metric space admits a bi-Lipschitz embedding into $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

1. Introduction

Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a mapping. Then f is called a *bi-Lipschitz embedding* if there are constants $A, B > 0$ such that

$$(1) \qquad A d_X(x, y) \le d_Y(f(x), f(y)) \le B d_X(x, y) \text{ for all } x, y \in X.$$

If f is a bi-Lipschitz embedding, then the *distortion* of f is defined to be the infimum of $\frac{B}{A}$ over all constants $A, B > 0$ for which (1) holds.

A metric space (X, d) is said to have a *doubling constant* $K \ge 1$ if for every $r > 0$ every closed ball in X of radius r can be covered by at most K closed balls of radius $\frac{r}{2}$. By a closed ball of radius r we mean a set of the form $B(x, r) = \{y \in X : d(y, x) \le r\}$, where $x \in X$ is the center of $B(x, r)$. The space X is called *doubling* if it has a doubling constant K for some $K \ge 1$. Note that doubling metric spaces are separable.

If (X, d) is a metric space and $\alpha \in (0, 1)$, then d^α is clearly also a metric on X and the space (X, d^α) is called the *α-snowflaked version* of (X, d). Note that (X, d) is doubling if and only if (X, d^α) is doubling (possibly with a different doubling constant).

An important open problem in embedding theory is to characterize intrinsically those metric spaces that admit a bi-Lipschitz embedding into $\mathbb{R}^n$ for some $n \in \mathbb{N}$ (we will always consider the Euclidean norm and metric on $\mathbb{R}^n$). It is easy to see that if a metric space admits a bi-Lipschitz embedding into $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then it must be doubling. It is known that the converse does not hold. For example, the 3-dimensional Heisenberg group with its Carnot metric is doubling but does not admit a bi-Lipschitz embedding into $\mathbb{R}^n$ for any $n \in \mathbb{N}$ (see [Se, Theorem 7.1]). However, Assouad [As, Proposition 2.6] proved the following fundamental theorem.

Theorem 1.1 (Assouad, 1983). *Let (X, d) be a doubling metric space and $\alpha \in (0, 1)$. Then (X, d^α) admits a bi-Lipschitz embedding into $\mathbb{R}^n$ for some $n \in \mathbb{N}$.*

Assouad's proof of Theorem 1.1, which we will present here, actually gives the following stronger quantitative statement.

Theorem 1.2 (Quantitative version of Assouad's theorem). *For every $K \ge 1$ and $\alpha \in (0, 1)$, there is an $N = N(K, \alpha) \in \mathbb{N}$ and $D = D(K, \alpha) \ge 1$ such that for every metric space (X, d) with a doubling constant K, the space (X, d^α) admits a bi-Lipschitz embedding into $\mathbb{R}^N$ with distortion at most D.*

Let us mention that in the original paper of Assouad [As], Theorem 1.1 is stated for metric spaces of finite Assouad dimension instead of for doubling metric spaces. Let (X, d) be a metric space. The *Assouad dimension* of X (called the *metric*

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dimension in [As]) is the infimum of those $\beta \geq 0$ for which there is $C > 0$ such that for every $0 < a < b$, for every $Y \subset X$ such that $d(x, y) > a$ whenever $x, y \in Y, x \neq y$, and for every $Z \subset X$ such that $\operatorname{diam} Z \leq b$, we have $|Y \cap Z| \leq C \left(\frac{b}{a}\right)^{\beta}$ (if M is a set, we denote by $|M|$ the cardinality of M). However, it is not difficult to prove that X is of finite Assouad dimension if and only if it is doubling. It is actually the value of the doubling constant, and not so much the Assouad dimension, that is relevant to the proof of Assouad's theorem, and so it seems more natural to state the theorem in the present form.

The purpose of this survey is to present in detail the proof of Theorem 1.2. In Section 2 we recall the notion of the tensor product of Hilbert spaces, which will be used in the proof. The proof itself is presented in Section 3. In Section 4, we discuss some questions concerning the dimension of the receiving space $\mathbb{R}^N$ in Theorem 1.2.

2. Tensor products of Hilbert spaces

In this section, we briefly recall the notion of the tensor product of Hilbert spaces, which will be used in the proof of Theorem 1.2. Those who are familiar with tensor products can skip this section.

Let us first describe the algebraic tensor product of linear spaces. Let V, W be linear spaces over $\mathbb{R}$. We denote by $\Lambda(V \times W)$ the set of all formal finite linear combinations of members of the Cartesian product $V \times W$, that is, the set of all expressions of the form $\sum_{i=1}^{n} a_i(e_i, f_i)$, where $a_i \in \mathbb{R}, e_i \in V, f_i \in W, i = 1, \ldots, n$, and $n \in \mathbb{N}$. We identify $\sum_{i=1}^{n} a_i(e_i, f_i)$ and $\sum_{i=1}^{n} a_{\pi(i)}(e_{\pi(i)}, f_{\pi(i)})$ for any permutation π of $\{1, \ldots, n\}$ and we also identify $\sum_{i=1}^{n+1} a_i(e_i, f_i)$ and $\sum_{i=1}^{n} a_i(e_i, f_i)$ if $a_{n+1} = 0$. We make $\Lambda(V \times W)$ into a linear space by defining

$$a\left(\sum_{i=1}^{n} a_i(e_i, f_i)\right) = \sum_{i=1}^{n} a a_i(e_i, f_i)$$

and

$$\sum_{i=1}^{n} a_i(e_i, f_i) + \sum_{i=1}^{n} b_i(e_i, f_i) = \sum_{i=1}^{n} (a_i + b_i)(e_i, f_i).$$

Furthermore, we denote by $\Lambda_0(V \times W)$ the linear subspace of $\Lambda(V \times W)$ generated by the elements of the form

$$(a_1 e_1 + a_2 e_2, b_1 f_1 + b_2 f_2) - \sum_{i,j=1}^{2} a_i b_j (e_i, f_j).$$

The *algebraic tensor product* of V and W, denoted by $V \otimes W$, is the linear quotient space $\Lambda(V \times W)/\Lambda_0(V \times W)$. Elements of $V \otimes W$ are called *tensors*. We denote by $e \otimes f$ the tensor containing (e, f), that is, the equivalence class $(e, f) + \Lambda_0(V \times W)$. So any tensor from $V \otimes W$ can be written as $\sum_{i=1}^{n} a_i e_i \otimes f_i$, where $a_i \in \mathbb{R}, e_i \in V, f_i \in W$ and $n \in \mathbb{N}$. The purpose of taking the quotient is that now we have

$$(a_1 e_1 + a_2 e_2) \otimes (b_1 f_1 + b_2 f_2) = \sum_{i,j=1}^{2} a_i b_j e_i \otimes f_j.$$

It is not hard to show that if $(e_i)_{i \in \Gamma_1}$ is a basis of V and $(f_j)_{j \in \Gamma_2}$ is a basis of W, then $(e_i \otimes f_j)_{(i,j) \in \Gamma_1 \times \Gamma_2}$ is a basis of $V \otimes W$. In particular, if V and W are finite dimensional, then $\dim(V \otimes W) = \dim V \dim W$.

Now, let H_1, H_2 be real Hilbert spaces with inner products $\langle ., . \rangle_1, \langle ., . \rangle_2$ and norms $\|.\|_1, \|.\|_2$ respectively. We define an inner product on the algebraic tensor product $H_1 \otimes H_2$ by setting $\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle e_1, e_2 \rangle_1 \langle f_1, f_2 \rangle_2$ for all $e_1, e_2 \in H_1, f_1, f_2 \in H_2$, and by extending bilinearly to all of $H_1 \otimes H_2$. It is of course necessary to check that $\langle .,. \rangle$ is well-defined and that it is indeed an inner product. As usual, the inner product $\langle .,. \rangle$ gives rise to a norm on $H_1 \otimes H_2$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in H_1 \otimes H_2$. The completion of $H_1 \otimes H_2$ under this norm, which is of course a Hilbert space, is called the *tensor product* of H_1 and H_2 and is also denoted by $H_1 \otimes H_2$ (from now on, we will always use this symbol for tensor products of Hilbert spaces, so no confusion should arise). Note that for any $e \in H_1, f \in H_2$ we have $\|e \otimes f\| = \|e\|_1 \|f\|_2$. Note also that if H_1 and H_2 are finite dimensional, then their algebraic tensor product is also finite dimensional and so the completion leaves the space unchanged. Hence in this case $\dim(H_1 \otimes H_2) = \dim H_1 \dim H_2$.

3. Proof of Theorem 1.2

Let us prove Theorem 1.2. We will basically follow the lines of the original proof of Assouad [As] (see also [He, Theorem 12.2] for an exposition in English). We will use the following lemma.

Lemma 3.1. *Let $\alpha, \tau \in (0,1)$, $A, B > 0$ and $m \in \mathbb{N}$. Then there is an $N \in \mathbb{N}$ and $D \geq 1$ such that if (X, d) is a metric space and there are mappings $\varphi_i \colon X \to \mathbb{R}^m$, $i \in \mathbb{Z}$, satisfying*

1. $\|\varphi_i(s) - \varphi_i(t)\| \geq A$ *if* $\tau^{i+1} < d(s,t) \leq \tau^i$,
2. $\|\varphi_i(s) - \varphi_i(t)\| \leq B \min\{\tau^{-i} d(s,t), 1\}$ *for all* $s, t \in X$,

then (X, d^α) admits a bi-Lipschitz embedding into $\mathbb{R}^N$ with distortion at most D.

Proof. Let (X, d) be a metric space and suppose that there are mappings $\varphi_i \colon X \to \mathbb{R}^m$, $i \in \mathbb{Z}$, satisfying the conditions (1) and (2). We will also work with the space $\mathbb{R}^{2n}$, where $n \in \mathbb{N}$ will be chosen later. Let $e_1, \ldots, e_{2n}$ be an orthonormal basis of $\mathbb{R}^{2n}$ (for example the canonical basis) and extend the sequence (e_i) $2n$-periodically to all of $\mathbb{Z}$, that is, $e_{i+2n} = e_i$ for every $i \in \mathbb{Z}$. Also, fix an arbitrary $s_0 \in X$.

We define a mapping $f \colon X \to \mathbb{R}^m \otimes \mathbb{R}^{2n}$ by

$$f(s) = \sum_{i \in \mathbb{Z}} \tau^{i\alpha} (\varphi_i(s) - \varphi_i(s_0)) \otimes e_i.$$

(By $\mathbb{R}^m \otimes \mathbb{R}^{2n}$ we mean the tensor product of the Hilbert spaces $\mathbb{R}^m$ and $\mathbb{R}^{2n}$ as described in Section 2. It is linearly isometric to $\mathbb{R}^{2mn}$.) The convergence of the series will follow from the first estimate bellow. Let us show that for large enough n the mapping f is a bi-Lipschitz embedding of (X, d^α) into $\mathbb{R}^m \otimes \mathbb{R}^{2n}$.

Let $s, t \in X, s \neq t$, and let $k \in \mathbb{Z}$ be such that $\tau^{k+1} < d(s,t) \leq \tau^k$. Let us first estimate $\|f(s) - f(t)\|$ from above. We have

$$\begin{aligned}
\|f(s) - f(t)\| &\leq \sum_{i>k} \tau^{i\alpha} \|\varphi_i(s) - \varphi_i(t)\| + \sum_{i \leq k} \tau^{i\alpha} \|\varphi_i(s) - \varphi_i(t)\| \\
&\leq \sum_{i>k} \tau^{i\alpha} B + \sum_{i \leq k} \tau^{i\alpha} B \tau^{-i} d(s,t) \\
&= B\tau^{(k+1)\alpha} \sum_{i=0}^{\infty} \tau^{i\alpha} + B d(s,t) \tau^{k(\alpha-1)} \sum_{i=0}^{\infty} \tau^{i(1-\alpha)} \\
&= B\tau^{(k+1)\alpha} \frac{1}{1-\tau^\alpha} + B d(s,t) \tau^{k(\alpha-1)} \frac{1}{1-\tau^{1-\alpha}} \\
&\leq B d(s,t)^\alpha \frac{1}{1-\tau^\alpha} + B d(s,t) d(s,t)^{\alpha-1} \frac{1}{1-\tau^{1-\alpha}} \\
&= B \left(\frac{1}{1-\tau^\alpha} + \frac{1}{1-\tau^{1-\alpha}} \right) d(s,t)^\alpha.
\end{aligned}$$

Note that no restriction on n was needed in this estimate.

Now, let us estimate $\|f(s)-f(t)\|$ from bellow. We have

$$\|f(s)-f(t)\| \geq \left\| \sum_{k-n<i\leq k+n} \tau^{i\alpha}(\varphi_i(s)-\varphi_i(t))\otimes e_i \right\| - \sum_{i>k+n} \tau^{i\alpha}\|\varphi_i(s)-\varphi_i(t)\| - \sum_{i\leq k-n} \tau^{i\alpha}\|\varphi_i(s)-\varphi_i(t)\|.$$

For the first sum we have

$$\left\| \sum_{k-n<i\leq k+n} \tau^{i\alpha}(\varphi_i(s)-\varphi_i(t))\otimes e_i \right\| \geq \tau^{k\alpha}\|\varphi_k(s)-\varphi_k(t)\| \geq \tau^{k\alpha}A \geq d(s,t)^\alpha A,$$

where the first inequality holds since the summands on the left hand side are mutually orthogonal. The second sum satisfies

$$\begin{aligned}\sum_{i>k+n} \tau^{i\alpha}\|\varphi_i(s)-\varphi_i(t)\| &\leq \sum_{i>k+n}\tau^{i\alpha}B = B\tau^{(k+n+1)\alpha}\sum_{i=0}^{\infty}\tau^{i\alpha} \\ &= B\tau^{(k+n+1)\alpha}\frac{1}{1-\tau^\alpha} \leq Bd(s,t)^\alpha\frac{\tau^{n\alpha}}{1-\tau^\alpha},\end{aligned}$$

and for the last sum we have

$$\begin{aligned}\sum_{i\leq k-n} \tau^{i\alpha}\|\varphi_i(s)-\varphi_i(t)\| &\leq \sum_{i\leq k-n}\tau^{i\alpha}B\tau^{-i}d(s,t) = Bd(s,t)\tau^{(k-n)(\alpha-1)}\sum_{i=0}^{\infty}\tau^{i(1-\alpha)} \\ &= Bd(s,t)\tau^{(k-n)(\alpha-1)}\frac{1}{1-\tau^{1-\alpha}} \leq Bd(s,t)^\alpha\frac{\tau^{n(1-\alpha)}}{1-\tau^{1-\alpha}}.\end{aligned}$$

Hence we obtain

$$\|f(s)-f(t)\| \geq \left(A - B\left(\frac{\tau^{n\alpha}}{1-\tau^\alpha}+\frac{\tau^{n(1-\alpha)}}{1-\tau^{1-\alpha}}\right)\right)d(s,t)^\alpha.$$

Now if n is large enough so that the constant on the right hand side is positive (which depends only on α, τ, A and B), then the mapping f is a bi-Lipschitz embedding of (X, d^α) into $\mathbb{R}^m \otimes \mathbb{R}^{2n}$ and both the dimension of the target space and the distortion of f depend only on α, τ, A, B and m. □

Proof of Theorem 1.2. Let $K \geq 1$ and fix an arbitrary $\tau \in (0,1)$. Let (X,d) be a metric space with a doubling constant K and let $i \in \mathbb{Z}$. We will construct a mapping $\varphi = \varphi_i \colon X \to \mathbb{R}^m$ for some $m \in \mathbb{N}$ such that the conditions (1) and (2) in Lemma 3.1 will be satisfied for some $A, B > 0$, and A, B and m will depend only on K and our choice of τ. Lemma 3.1 will then complete the proof of Theorem 1.2.

Let $c = \frac{1}{4}\tau^{i+1}$ and take a c-net Y in X. By a c-net we mean a maximal subset of X such that all pairs of its distinct points have distance at least c. By Zorn's lemma, such a set exists. It is then clear that for every $y \in Y$ we have

$$\left|\left\{z \in Y : d(z,y) \leq \left(\frac{4}{\tau}+4\right)c\right\}\right| \leq m,$$

where $m \in \mathbb{N}$ depends only on the doubling constant K and the choice of τ (we can take any $m \geq K^{2+\log_2\left(\frac{4}{\tau}+4\right)}$). Let $k\colon Y \to \{1,\dots,m\}$ be an $(m, (\frac{4}{\tau}+4)c)$-colouring of Y, that is, $k(y) \neq k(y')$ if $y, y' \in Y, y \neq y'$, and $d(y,y') \leq (\frac{4}{\tau}+4)c$. Such a mapping clearly exists. Indeed, since Y is clearly countable, we can make it into a sequence (y_j) and define $k(y_j)$ inductively by choosing a value from $\{1,\dots,m\}$ not taken by those y_l for $l < j$ for which $d(y_l, y_j) \leq (\frac{4}{\tau}+4)c$. Since there are at most $m-1$ such y_l, this is always possible.

Let $e_1, \dots, e_m$ be an orthonormal basis of $\mathbb{R}^m$. We define $\varphi\colon X \to \mathbb{R}^m$ by

$$\varphi(s) = \sum_{y \in Y} g_y(s) e_{k(y)},$$

where

$$g_y(s) = \frac{1}{2c} \max\{2c - d(s,y), 0\}.$$

Let us verify that φ is the desired mapping.

First, if $s \in X$, let $B_s = \{y \in Y : g_y(s) \neq 0\} = \{y \in Y : d(y,s) < 2c\}$. Then clearly $|B_s| \leq m$. This in particular shows that the sum in the definition of $\varphi(s)$ is in fact finite, hence convergent. Let $s, t \in X$. It is clear that for every $y \in Y$ we have

$$|g_y(s) - g_y(t)| \leq \frac{1}{2c} d(s,t) = \frac{2}{\tau} \tau^{-i} d(s,t),$$

and therefore

$$\|\varphi(s) - \varphi(t)\| \leq \sum_{y \in B_s \cup B_t} |g_y(s) - g_y(t)| \leq 2m \frac{2}{\tau} \tau^{-i} d(s,t) = \frac{4m}{\tau} \tau^{-i} d(s,t).$$

Furthermore, we have

$$\begin{aligned}\|\varphi(s) - \varphi(t)\| \leq \|\varphi(s)\| + \|\varphi(t)\| &= \left\| \sum_{y \in B_s} g_y(s) e_{k(y)} \right\| + \left\| \sum_{y \in B_t} g_y(t) e_{k(y)} \right\| \\ &\leq 2m \leq \frac{4m}{\tau}.\end{aligned}$$

Hence

$$\|\varphi(s) - \varphi(t)\| \leq \frac{4m}{\tau} \min\{\tau^{-i} d(s,t), 1\},$$

and therefore the condition (2) in Lemma 3.1 is satisfied with $B = \frac{4m}{\tau}$.

Now, let $s, t \in X$ be such that $4c = \tau^{i+1} < d(s,t) \leq \tau^i = \frac{4}{\tau} c$. Then $B_s \cap B_t = \emptyset$ and the vectors $e_{k(y)}$ for $y \in B_s \cup B_t$ are mutually orthogonal, and therefore

$$\|\varphi(s) - \varphi(t)\|^2 = \sum_{y \in B_s} |g_y(s)|^2 + \sum_{y \in B_t} |g_y(t)|^2.$$

Since Y is a c-net, there is a $y \in Y$ such that $d(y,s) < c$. Then $g_y(s) \geq \frac{1}{2}$, and therefore $\|\varphi(s) - \varphi(t)\| \geq \frac{1}{2}$. Hence the condition (1) in Lemma 3.1 is satisfied with $A = \frac{1}{2}$ and the proof is complete. □

4. The dimension of the receiving Euclidean space

Let $K \geq 2$ and $\alpha \in (0,1)$ be fixed. Let us inspect the above proof of Theorem 1.2 to see how large the dimension $N(K,\alpha)$ it gives. We are interested in an estimate from below. At the beginning of the proof we choose an arbitrary $\tau \in (0,1)$. Then we take $m \in \mathbb{N}$ such that $m \geq K^{2+\log_2\left(\frac{4}{\tau}+4\right)}$, and $A = \frac{1}{2}$ and $B = \frac{4m}{\tau}$. The dimension $N(K,\alpha)$ is then equal to $2mn$, where $n \in \mathbb{N}$ is such that

$$\frac{\tau^{n\alpha}}{1-\tau^\alpha} + \frac{\tau^{n(1-\alpha)}}{1-\tau^{1-\alpha}} < \frac{A}{B} = \frac{\tau}{8m}.$$

Then we must have

$$\frac{\tau^{n\alpha}}{1-\tau^\alpha} < \frac{\tau}{8m},$$

and therefore

$$\begin{aligned} n &> \frac{\log_2\left(\frac{\tau(1-\tau^\alpha)}{8m}\right)}{\alpha\log_2\tau} = \frac{\log_2\left(\frac{8m}{\tau(1-\tau^\alpha)}\right)}{\alpha\log_2\frac{1}{\tau}} = \frac{\log_2 8+\log_2 m-\log_2(\tau(1-\tau^\alpha))}{\alpha\log_2\frac{1}{\tau}} \\ &\geq \frac{\log_2 m}{\alpha\log_2\frac{1}{\tau}} \geq \frac{\log_2 K^{2+\log_2\left(\frac{4}{\tau}+4\right)}}{\alpha\log_2\frac{1}{\tau}} = \frac{\left(2+\log_2\left(\frac{4}{\tau}+4\right)\right)\log_2 K}{\alpha\log_2\frac{1}{\tau}} \\ &\geq \frac{\log_2\frac{1}{\tau}\log_2 K}{\alpha\log_2\frac{1}{\tau}} = \frac{\log_2 K}{\alpha}. \end{aligned}$$

Similarly, we must have

$$\frac{\tau^{n(1-\alpha)}}{1-\tau^{1-\alpha}} < \frac{\tau}{8m},$$

and therefore

$$n > \frac{\log_2 K}{1-\alpha}.$$

Hence

$$n > \log_2 K \max\left\{\frac{1}{\alpha}, \frac{1}{1-\alpha}\right\}.$$

It follows that no matter which $\tau \in (0,1)$ we choose at the beginning of the proof we obtain

$$N(K,\alpha) = 2mn \geq 2K^4\log_2 K\max\left\{\frac{1}{\alpha}, \frac{1}{1-\alpha}\right\}$$

(here we used the fact that $m \geq K^{2+\log_2\left(\frac{4}{\tau}+4\right)} \geq K^{2+\log_2 4} = K^4$). In particular, the construction gives $N(K,\alpha) \to \infty$ as $\alpha \to 0$ and also as $\alpha \to 1$. Is this necessary?

To answer this question, we can start by trying to optimize the constants that come into the construction. For example, it is not clear at first sight whether we can take some $m < K^{2+\log_2\left(\frac{4}{\tau}+4\right)}$ that would work as well. However, let us take a different point of view. In this context, the notion of Assouad dimension introduced after Theorem 1.2 proves useful. Let us denote the Assouad dimension of a metric space (X,d) by $\dim_A(X,d)$. It is not difficult to prove the following facts (see also [As]).

- $\dim_A(\mathbb{R}^n) = n$ for every $n \in \mathbb{N}$.
- $\dim_A(X,d^\alpha) = \frac{1}{\alpha}\dim_A(X,d)$ for every $\alpha \in (0,1)$.
- If (X,d) admits a bi-Lipschitz embedding into a metric space (Y,δ), then $\dim_A(X,d) \leq \dim_A(Y,\delta)$.

It follows that if (X,d) is a doubling metric space and $\alpha \in (0,1)$, then in order to have a bi-Lipschitz embedding of (X,d^α) into $\mathbb{R}^n$ we must have $n \geq \frac{1}{\alpha}\dim_A(X,d)$. In particular, in Theorem 1.2 we must have $N(K,\alpha) \to \infty$ as $\alpha \to 0$ for any $K \geq 2$ (by taking e.g. $X = \mathbb{R}$). However, note that if $\alpha \in (b,1)$ for some $b \in (0,1)$, then this method does not show any obstruction for having a bi-Lipschitz embedding of (X,d^α) into $\mathbb{R}^n$ for some $n \in \mathbb{N}$ independent of α. It turns out that this is not accidental. Indeed, Naor and Neiman [NN, Theorem 1.2] proved the following theorem.

Theorem 4.1 (Naor, Neiman, 2012)**.** *For every $K \geq 1$ there is an $N = N(K) \in \mathbb{N}$ and for every $K \geq 1$ and $\alpha \in (\frac{1}{2},1)$ there is a $D = D(K,\alpha) \geq 1$ such that for every metric space (X,d) with a doubling constant K, the space (X,d^α) admits a bi-Lipschitz embedding into $\mathbb{R}^N$ with distortion at most D.*

Note that the theorem holds true if we replace $\frac{1}{2}$ with any fixed constant $b \in (0,1)$. The point is to have α bounded away from 0. Let us mention that the proof in [NN] gives the estimates

$$N(K) \leq C \log K \text{ and } D(K,\alpha) \leq C \left(\frac{\log K}{1-\alpha} \right)^2,$$

where $C > 0$ is some absolute constant. We will not discuss the proof of Theorem 4.1 here. Let us just say that the proof of Naor and Neiman is probabilistic. Later, David and Snipes [DS] found a non-probabilistic proof of Theorem 4.1 based on the original construction of Assouad.

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