POINCARÉ TYPE INEQUALITIES AND NON-EMBEDDABILITIES: GROSS TRICK AND SPHERE EQUIVALENCE

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Abstract. This report describes a rough sketch of proofs and explains the motivation of main results in the paper "Sphere equivalence, Banach expanders, and extrapolation" (in Int. Math. Res. Notices) [Mim14] by the author. Specially, we indicate some potentially use of group theory, which we call "the Gross trick", to study metric embeddings of general graphs.

1. Motivations

First we give our notation. Unless stating, we always assume the following:

- $\Gamma = (V, E)$ is a finite connected undirected graph, possibly with multiple edges and self-loops (here E is the set of oriented edges). Γ is a metric space with the path metric d_Γ (namely, $d_\Gamma(v, w)$ is the shortest length of a path connecting v and w, and set $d_\Gamma(v, v) = 0$), and $\mathrm{diam}(\Gamma)$ means the diameter (the length of largest distance).
- For $v \in V$, $\deg(v)$, the degree of v, is the number of edges which starts at v. Note that a *self-loop contributes twice* to the degree of the vertex. $\Delta(\Gamma)$ is the maximal degree $\max_{v\in V} \deg(v)$ of Γ.
- $\{\Gamma_n = (V_n, E_n)\}_n$ is a sequence of finite graphs.
- (X, p) is a pair of a Banach space X and an exponent p. *We always assume that* $p \in [1, \infty)$ (in particular, *p is always assumed to be finite.*)
- Y is also used for a Banach space. q is also used for an exponent in $[1, \infty)$.
- For $r \in [1, \infty]$ and $k \geq 1$, ℓ_r^k stands for the real ℓ_r-space of dimension k. ℓ_r means the real ℓ_r-space over an infinite countable set.
- In this report, $\tilde{X}_{(p)}$ means $\ell_p(\mathbb{N}, X)$.
- For X, $S(X)$ is the unit sphere of X.
- In a metric space L and $A, B \subseteq L$, $\mathrm{dist}(A, B)$ means the distance, namely, $\inf\{d_L(a, b) : a \in A, b \in B\}$.
- $a \precsim b$ for two nonnegative functions from the same parameter set $\mathcal{T}$ means that there exists $C > 0$ *independent of* $t \in \mathcal{T}$ such that for any $t \in \mathcal{T}$, $a(t) \leq Cb(t)$. $a \asymp b$ means both $a \precsim b$ and $a \succsim b$ hold. $a \precsim_q b$ if parameter set $\mathcal{T}$ has variable q and $C = C_q$ may depend on q.
- We write $a \precnsim b$ if $a \precsim b$ holds but $a \succsim b$ fails to be true.

1.1. **Classical spectral gaps.** Here assume that Γ is k-regular (that means, $\deg(v) = k$ for all $v \in V$). Then the (nonnormalized) Laplacian $L(\Gamma) := kI_V - A(\Gamma)$, $A(\Gamma)$ being the adjacency matrix (the matrix $(a_{v,w})_{v,w}$ where $a_{v,w}$ is the number of edges connecting v

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and w, counting self-loop twice), is a positive operator and has eigenvalues $0 = \lambda_0(\Gamma) < \lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \cdots \leq \lambda_{|V|}(\Gamma)$. This $\lambda_1(\Gamma)$ is the *classical spectral gap* of Γ. This has a *Rayleigh quotient formula*:

$$\lambda_1(\Gamma) = \frac{1}{2} \inf_{f\colon V\to\mathbb{R}} \frac{\sum_{v\in V}\sum_{e=(v,w)\in E}|f(w)-f(v)|^2}{\sum_{v\in V}|f(v)-m(f)|^2}. \quad \cdots\ (*)$$

Here $m(f) := \sum_{v\in V} f(v)/|V|$ and f runs over all nonconstant maps.

1.2. **Banach spectral gaps.** The point in $(*)$ is that $\mathbb{R}$ has a metric and a mean structures.

Definition 1.1. For (X,p), define the the (X,p)-*spectral gap* of Γ by

$$\lambda_1(\Gamma; X, p) := \frac{1}{2} \inf_{f\colon V\to X} \frac{\sum_{v\in V}\sum_{e=(v,w)\in E}\|f(w)-f(v)\|^p}{\sum_{v\in V}\|f(v)-m(f)\|^p}. \quad \cdots\ (**)$$

Here $m(f) := \sum_{v\in V} f(v)/|V|$ and f runs over all nonconstant maps.

Example 1.2. $\lambda_1(\Gamma) = \lambda_1(\Gamma;\mathbb{R},2) = \lambda_1(\Gamma;\ell_2,2)$ (the latter equality is by Lemma 1.3). It is known that $\lambda_1(\Gamma;\mathbb{R},1)$ is proportional to $h(\Gamma)$, the (edge-)isoperimetric constant (also known as (nonnormalized) Cheeger constant) of Γ, see [Chu97, Theorem 2.5]. Here $h(\Gamma)$ is defined as $\inf\{|E(A, V\setminus A)|/|A| : 0 < |A| \leq |V|/2\}$, where $E(A, V\setminus A) := \{e = (v,w) \in E : v \in A, w \in V \setminus A\}$.

We note that Mendel and Naor [MN12] have explicitly introduced the notion of *nonlinear spectral gaps* (for the more general case where X is a metric space) and studied that in detail.

1.3. **Poincaré-type inequality.** $(**)$ is equivalent to saying the following:

$$\forall f\colon V\to X,\quad \sum_{v\in V}\|f(v)-m(f)\|^p \leq \frac{1}{\lambda_1(\Gamma;X,p)}\frac{1}{2}\sum_{v\in V}\sum_{e=(v,w)\in E}\|f(w)-f(v)\|^p. \quad \cdots\ (***)$$

This bounds the "p-variance" from below by the "p-energy" in a rough sense.

Lemma 1.3. (1) *If Y is a subspace of X, then $\lambda_1(\Gamma;Y,p) \geq \lambda_1(\Gamma;X,p)$.*
(2) $\lambda_1(\Gamma;X,p) = \lambda_1(\Gamma;\tilde{X}_{(p)},p)$.
In particular, $\lambda_1(\Gamma;\mathbb{R},p) = \lambda_1(\Gamma;\ell_p,p)$.

Proof. (1) is trivial. For (2), $\geq$ is from (1). To get $\leq$, integrate $(***)$ over $\mathbb{N}$. □

1.4. **Banach expanders.**

Definition 1.4. A sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ is called (X,p)-*anders* if the following three conditions are satisfied:

(*i*) $\sup_n \Delta(\Gamma_n) < \infty$;
(*ii*) $\lim_{n\to\infty} \mathrm{diam}(\Gamma_n) = \infty$;
(*iii*) There exists $\epsilon > 0$ such that $\inf_n \lambda_1(\Gamma_n; X, p) \geq \epsilon$.

(Classical) expanders equal $(\mathbb{R},2)$-anders, which also equal $(\mathbb{R},p)$-anders for all p by Matoušek's extrapolation (Theorem 1.16). By Lemma 1.3, they are also equal to (ℓ_p,p)-anders.

1.5. **Who cares? 1: coarse embeddings.**

Definition 1.5. Let (Λ, d_Λ) be a metric space. We say $f\colon \Lambda \to X$ is a *coarse embedding* if there exist a nondecreasing $\rho_-\rho_+\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\lim_{t\to+\infty}\rho_-(t) = +\infty$ such that for any $v, w \in \Lambda$,

$$\rho_-(d_\Lambda(v,w)) \leq \|f(v) - f(w)\|_X \leq \rho_+(d_\Lambda(v,w)).$$

This (ρ_-, ρ_+) is called a *control pair.*

For $\{\Gamma_n\}_n$ with $\lim_{n\to\infty}\operatorname{diam}(\Gamma_n) = \infty$, define a *coarse disjoint union* $\coprod_n \Gamma_n$ to be an (infinite) metric space $(\coprod_n \Gamma_n, d)$ whose point set is $\bigsqcup_n V_n$ and whose metric satisfies:

- For every n, $d\,|_{V_n\times V_n} = d_n$, where d_n denotes the original metric on Γ_n.
- For $n \neq m$, $\operatorname{dist}(V_n, V_m) \geq \operatorname{diam}(\Gamma_n) + \operatorname{diam}(\Gamma_m)$.

Theorem 1.6 (Matoušek, Gromov, Higson, et al.)**.** *Let $\{\Gamma_n\}_n$ be (X,p)-anders for some p. Then $\coprod_n \Gamma_n$ does not admit coarse embeddings into X.*

Proof. Take $\epsilon > 0$ in Definition 1.4 and $K := \epsilon^{-1}$. Suppose, in contrary, that $f\colon \coprod_n \Gamma_n \to X$ be a coarse embedding with control pair (ρ_-, ρ_+). Set $f_n := f|_{V_n}$. For considering each f_n, we may assume $m(f_n) = 0$. Then by $(***)$,

$$\begin{aligned}\frac{1}{|V_n|}\sum_{v\in V_n}\|f_n(v)\|^p &\leq \frac{1}{2|V_n|}K\sum_{v\in V_n}\sum_{e=(v,w)\in E_n}\|f_n(w) - f_n(v)\|^p\\ &\leq K\Delta(\Gamma_n)\rho_+(1)^p.\end{aligned}$$

Therefore, by letting $M = (2K\sup_n \Delta(\Gamma_n))^{1/p}\rho_+(1)$ (independent on n), we have that at least half of $v \in V_n$ satisfies $\|f_n(v)\| \leq M$. Because $\operatorname{diam}(\Gamma_n) \to \infty$, this contradicts that $\lim_{t\to+\infty}\rho_-(t) = +\infty$. $\square$

Remark 1.7. Recently Arzhantseva and Tessera [AT14] prove the following:

Theorem 1.8 ([AT14])**.** *There exists $\{\Gamma_n\}_n$ such that*

(i) $\sup_n \Delta(\Gamma_n) < \infty$;
(ii) $\coprod_n \Gamma_n$ *does not admit coarse embeddings into* ℓ_2;
(iii) but $\coprod_n \Gamma_n$ *does not admit weak embeddings of any expanders into itself.*

Here a sequence $\{\Lambda_m\}_m$ of finite graphs is said to admit a weak embedding into a metric space Z if there exist $K > 0$ and K-Lipschitz maps $f_m\colon \Lambda_m \to Z$ such that $\lim_{m\to\infty}\sup_{v\in V(\Lambda_m)}|f_m^{-1}(f_m(v))|/|\Lambda_m| = 0.$

This shows that expanders are not the only obstruction to admitting coarse embeddings into ℓ_2. Their proof of *(ii)* employs some sorts of *relative Poincaré-type inequalities.*

1.6. **Who cares? 2: distortions.**

Definition 1.9. The *distortion* of Γ into X, denoted by $c_X(\Gamma)$ is defined by

$$c_X(\Gamma) := \inf\left\{C > 0 : \begin{array}{r}\exists f\colon V \to X, \exists r > 0 \text{ such that } \forall v,w \in V,\\ rd(v,w) \leq \|f(v) - f(w)\| \leq Crd(v,w)\end{array}\right\}.$$

We have $1 \leq c_{\ell_2^{|V|}}(\Gamma) \leq \operatorname{diam}(\Gamma)$. The latter estimate is obtained by the trivial embedding: $\Gamma \ni v \mapsto \delta_v \in \ell_2(V)$. Hence, by the Dvoretzy theorem, for infinite dimensional X, we have

$$1 \leq c_X(\Gamma) \precsim_X \operatorname{diam}(\Gamma).$$

Theorem 1.10 (Generalized Grigorchuk–Nowak inequality, see [GN12] and Theorem 2.3 of [Mim14])**.** *For any* $\epsilon \in (0,1)$,

$$c_X(\Gamma) \geq \frac{(1-\epsilon)^{1/p} r_\epsilon(\Gamma)}{2} \mathrm{diam}(\Gamma) \left(\frac{\lambda_1(\Gamma; X, p)}{\Delta(\Gamma)} \right)^{1/p}.$$

Here $r_\epsilon(\Gamma)$ *is defined as* $\inf\{\mathrm{diam}(A)/\mathrm{diam}(\Gamma) : |A| \geq \epsilon |V|\}$.

Theorem 1.11 (Special case of a generalized Jolissaint–Valette inequality, see [JV14] and Theorem 2.3 of [Mim14])**.** *Let* Γ *be a vertex-transitive graph (this means that the graph automorphism group acts* V *transitively). Then*

$$c_X(\Gamma) \geq 2^{-(p-1)/p} \mathrm{diam}(\Gamma) \left(\frac{\lambda_1(\Gamma; X, p)}{\Delta(\Gamma)} \right)^{1/p}.$$

Note that, as we will recall in Section 3, all Cayley graphs are vertex-transitive.

Corollary 1.12. *For infinite dimensional* X*, assume* $\{\Gamma_n\}_n$ *be* (X,p)*-anders for some* p*. Then* $c_X(\Gamma_n) \asymp_X \mathrm{diam}(\Gamma_n)$.

Proof. Note that $\{\Gamma_n\}_n$ is in particular a family of expanders (see (3) of Corollary 1.17) and is of (uniformly) exponential growth. If you do not know this fact, then this is deduced from the Matoušek extrapolation (Theorem 1.16) and Example 1.2 on isoperimetric constants.

Hence the conclusion follows from Theorem 1.10 and the discussion above. □

Lemma 1.13 (Austin's lemma [Aus11], see also in Lemma 2.7 in [Mim14])**.** *Let* $\{\Gamma_n\}_n$ *satisfy* $\mathrm{diam}(\Gamma_n) \nearrow \infty$ *(possibly with* $\sup_n \Delta(\Gamma_n) = \infty$*). Let* $\rho \colon \mathbb{R}_+ \nearrow \mathbb{R}_+$ *be a map with* $\lim_{t\to+\infty} \rho(t) = +\infty$ *which satisfies that* $\rho(t)/t$ *is nonincreasing for* t *large enough. Assume that for* n *large enough* $\frac{\mathrm{diam}(\Gamma_n)}{\rho(\mathrm{diam}(\Gamma_n))} \precnsim c_X(\Gamma_n)$ *hold. Then for any* $C>0$*,* (ρ, Ct) *is* not *a control pair of* $\coprod_n \Gamma_n$ *into* X.

Proof. Assume, in the contrary, that there exists a coarse embedding $f \colon \coprod_n \Gamma_n \to X$ such that

$$\rho(d(v,w)) \leq \|f(v) - f(w)\| C d(v,w), \quad v, w \in \coprod_n \Gamma_n$$

holds. Set $f_n := f|_{\Gamma_n} \colon V_n \to X$. We may assume, by rescaling, that f is a 1-Lipschitz map and that each f_n is biLipschitz. Then we have the following order inequalities.

$$\frac{\mathrm{diam}(\Gamma_n)}{\rho(\mathrm{diam}(\Gamma_n))} \precnsim c_X(\Gamma_n) \leq \|f_n^{-1}\|_{\mathrm{Lip}} \leq \max_{v\neq w \in V_n} \frac{d(v,w)}{\|f_n(v) - f_n(w)\|}$$
$$\precsim \max_{v\neq w \in V_n} \frac{d(v,w)}{\rho(d(v,w))} \precsim \frac{\mathrm{diam}(\Gamma_n)}{\rho(\mathrm{diam}(\Gamma_n))}.$$

This is a contradiction. □

Lemma 1.13, together with Corollary 1.12, gives an alternative proof of Theorem 1.6. Indeed, suppose, in contrary, that there exists a coarse embedding f of (X,p)-anders into X. By rescaling, we may assume that the control pair for f is (ρ, t) for some ρ (note that because $\coprod_n \Gamma_n$ is uniformly discrete, ρ_+ may be taken as linear function). By replacing ρ with a smaller proper function if necessary, we may also assume that $\rho(t)/t$ is nonincreasing for t large enough. Then Lemma 1.13 and Corollary 1.12 give the desired contradiction.

1.7. **Motivating problem.** A naive question on (X,p)-anders might be: *"Are any expanders are automatically (X,p)-anders for all (X,p)?"* The answer is *no*. Indeed, by the *Fréchet embedding*:

$$V_n \ni v \mapsto (d(v,w))_{w \in V_n},$$

Γ_n embeds isometrically into $\ell_\infty^{|V_n|}$. Thus if X has *trivial cotype*, then there exists a biLipschitz embedding of any $\coprod_n \Gamma_n$ into X. Here X is said to have *trivial cotype* if X contains uniformly isomorphic (in particular uniformly biLipschitz) copies of $\{\ell_\infty^n\}_n$.

The following question is a big open problem in this field:

Problem 1.14. *Are any expanders are automatically (X,p)-anders for all X of nontrivial cotype and for all p?*

In this report, we study the following two questions:

Problem 1.15. *For arbitrarily taken Γ,*
(a) estimate $\lambda_1(\Gamma; Y, p)$ from $\lambda_1(\Gamma; X, p)$;
(b) estimate $\lambda_1(\Gamma; X, q)$ from $\lambda_1(\Gamma; X, p)$.
In both cases, estimates may depend on $\Delta(\Gamma)$, but not *on $|\Gamma|$ itself.*

1.8. **previously known results.**
(b): Matoušek extrapolation

Theorem 1.16 ([Mat97])**.** (1) *For $p \in [1,2)$, $\lambda_1(\Gamma;\mathbb{R},2)^{p/2} \gtrsim_{\Delta(\Gamma),p} \lambda_1(\Gamma;\mathbb{R},p) \gtrsim_{\Delta(\Gamma),p} \lambda_1(\Gamma;\mathbb{R},2)$.*
(2) *For $p \in [2,\infty)$, $\lambda_1(\Gamma;\mathbb{R},p) \asymp_{\Delta(\Gamma),p} \lambda_1(\Gamma;\mathbb{R},2)^{p/2}$.*

Corollary 1.17. (1) *For any p, $\{\Gamma_n\}_n$ are expanders if and only if they are $(\mathbb{R},p)$-anders.*
(2) *Expanders do not admit coarse embeddings into ℓ_p for any p.*
(3) *For any (X,p), (X,p)-anders are (classical) expanders.*

Proof. (1) immediately follows. (2) is from Theorem 1.6. (3) follows from $X \supseteq \mathbb{R}$. ☐

(a): Pisier [Pis10]

The following definition is in [Pis10], which uses some idea of V. Lafforgue: X is said to be *uniformly curved* if $\lim_{\epsilon\to+0} D_X(\epsilon)= 0$ holds. Here $D_X(\epsilon)$ denote the infimum over those $D \in (0,\infty)$ such that for every $n \in \mathbb{N}$, every matrix $T = (t_{ij})_{i,j} \in \mathrm{M}_n(\mathbb{R})$ with

$$\|T\|_{\ell_2^n \to \ell_2^n} \le \epsilon \quad \text{and} \quad \|\mathrm{abs}(T)\|_{\ell_2^n \to \ell_2^n} \le 1,$$

where $\mathrm{abs}(T) = (|t_{ij}|)_{i,j}$ is the entrywise absolute value of T, satisfies that

$$\|T \otimes I_X\|_{\ell_2(n,X) \to \ell_2(n,X)} \le D.$$

Theorem 1.18 ([Pis10])**.** *Expanders are automatically $(X,2)$-anders for any uniformly curved Banach space X.*

Expamles of uniformly curved Banach spaces are ℓ_p, L_p, noncommutative L_p spaces, for $p \in (1,\infty)$, and more generally are given by complex interpolation theorey.

Remark 1.19. Pisier also showed in [Pis10] that uniformly curved Banah spaces are *superreflexive*, which is equivalent to admitting equivalent and uniformly convex norms. Recall that X is said to be *uniformly convex* if for any $\epsilon \in (0,2]$,

$$\sup\{\|x+y\|/2 : x,y \in S(X),\ \|x-y\| \ge \epsilon\} < 1.$$

We also mention that the existence of "special expanders", which have the expander property for a wider class of Banach spaces, is known independently by V. Lafforgue [Laf08] and Mendel–Naor [MN12]:

Theorem 1.20 ([Laf08], [MN12]). *There exist* (*explicitly constructed*) *expanders* $\{\Gamma_n\}_n$ *which are* $(X, 2)$*-anders for any* X *of nontrivial type.*

Here recall that X has *trivial type* if and only if X contains uniformly isomorphic copies of $\{\ell_1^n\}_n$.

2. Main results

2.1. Sphere equivalence and Ozawa's result. In [Mim14], the author call the following equivalence the *sphere equivalence*. This has been intensively studied for several years, and we refer the reader to Chapter 9 of [BL00].

Definition 2.1. X and Y are said to be *sphere equivalent*, written as $X \sim_S Y$, if there exists a uniform homeomorphism (, namely, a biuniformly continuous map) between $S(X)$ and $S(Y)$. We write $[Y]_S$ for the sphere equivalence class of Y.

If X and Y are isomorphic (in other words, if Y has an equivalent norm to that of X), then clearly $X \sim_S Y$. There, however, exist many nonisomorphic Banach spaces which are sphere equivalent.

Example 2.2. The sphere equivalence class of Hilbert spaces for instance contains the following:

- ℓ_p, L_p for any p: a uniform homeomophism is given by the *Mazur map*. For ℓ_p, the Mazur map is
$$M_{p,2} \colon S(\ell_p) \to S(\ell_2); \quad (a_i)_i \mapsto (\mathrm{sign}(a_i)|a_i|^{p/2})_i.$$
- Noncommutative L_p spaces associated with arbitrary von Neumann algebras [Ray02].
- Any Banach space of nontrivial cotype with unconditional basis [OS94].

Note that this sphere equivalence may go beyond superreflexivity; and moreover having nontrivial type. Indeed, the results mentioned above on (noncommutative) L_p spaces hold even for $p = 1$.

Example 2.3. Another example is given by complex interpolations (for a comprehensive treatise of complex interpolation, see a book [BL76]). Theorem 9.12 in [BL00] states that for a complex interpolation pair (X_0, X_1), if either X_0 or X_1 is uniformly convex, then any $0 < \theta < \theta' < 1$, $X_\theta \sim_S X_{\theta'}$. This result will be used for the proof of our main results.

On (a) of Problem 1.15, Ozawa [Oza04] made the first contribution.

Theorem 2.4 ([Oza04]). *If* $X \sim_S \ell_2$, *then expanders do not admit coarse embeddings into* X. *In fact, any expanders satisfy a weak form of* $(X, 1)$*-ander condition for such* X.

2.2. Main results. Here we exhibit main results in this report, extracted from [Mim14].

Theorem A (For more precise statement, see Theorem 4.1 in [Mim14]). *Assume* $X \sim_S Y$. *Then for any* $p \in [1, \infty)$, *and a sequence* $\{\Gamma_n\}_n$, $\{\Gamma_n\}_n$ *are* (X, p)*-anders if and only if they are* (Y, p)*-anders.*

More precisely, for a uniform homeomorphism $\phi \colon S(X) \to S(Y)$, *we may bound* $\lambda_1(\Gamma; X, p)$ *from below in terms of*

- $\lambda_1(\Gamma; Y, p)$;
- *the modulus of continuity of* ϕ;
- *and some constants depending on* p, $\Delta(\Gamma)$, *and the modulus of continuity of* ϕ^{-1}.

For instance, if ϕ *is* α*-Hölder continuous for some* $\alpha \in (0,1]$, *then we have*

$$\lambda_1(\Gamma; X, p) \gtrsim_{p,\Delta(\Gamma),M} \lambda_1(\Gamma; Y, p)^{1/\alpha}.$$

Here M *is a constant only depend on the modulus of continuity of* ϕ^{-1}.

Note that on the estimation above, the order of the estimate (for instance, the Hölder exponent if we have an estimate of such type) depends *only on* the modulus of continuity of ϕ. The modulus of continuity of the inverse map ϕ^{-1} appears only on postive scalar constant in our estimate.

Theorem B (Generalization of Matoušek's extrapolation)**.** *Let* $(\infty >)p, q > 1$. *Then for any* X *sphere equivalent to a uniformly convex Banach space, and a sequence* $\{\Gamma_n\}_n$, $\{\Gamma_n\}_n$ *are* (X, p)*-anders if and only if they are* (X, q)*-anders.*

Remark 2.5. We note that recently Naor, in Theorem 1.10 and Theorem 4.15 in [Nao14], has independently established similar results. Our approach is *group theoretic*, and different from his. In our proof, we introduce the *"Gross trick"*, see Section 6.

As byproducts to above Theorems A and B, and aforementioned works of Ozawa and Pisier; and Lafforgue and Mendel–Naor, we have the following corollaries.

Corollary C. *Any expanders are automatically* (X, p)*-anders for an* X *sphere equivalent to uniformly curved Banach space and for* $p \in (1, \infty)$. *If, moreover,* $X \in [\ell_2]_S$, *then the assertion above holds even for* $p = 1$.

In particular, for expanders $\{\Gamma_n\}_n$, *we have for such* X *of infinite dimension that*

$$c_X(\Gamma_n) \asymp_X \mathrm{diam}(\Gamma_n).$$

Corollary D. *The expanders constructed in Theorem* 1.20 *are* $(Y, 2)$*-anders for any* Y *sphere equivalent to a Banach space with nontrivial type.*

In particular, they do not admit coarse embedding into any such Y.

Note that, for instance, noncommutative L_1 spaces are examples of such Y with trivial type (though all expanders do not admit coarse embeddings to them by Theorem 2.4).

In the view of the results above, the following questions might be of importance.

Problem 2.6. (1) *Does the class of Banach spaces sphere equivalent to uniformly curved Banach spaces contain all superreflexive Banach spaces? Does it contain all Banach spaces of nontrivial type/nontrivial cotype?*

(2) *Does the class of Banach spaces sphere equivalent to Banach spaces of nontrivial type coincide with the class of all Banach spaces of nontrivial cotype?*

To the best of my knowledge, all of the problems above may be open.

Remark 2.7. On (2), one inclusion is verified from Corollary D and Subsection 1.7 (also, in [BL00], the authors of the book announced a result that the class of Banach spaces with trivial cotype is closed under the sphere equivalence). Hence, the true question in (2) is whether the sphere equivalence class above contain all Banach spaces of nontrivial cotype.

Remark 2.8. There is also a notion of "*ball equivalence*" (namely, the unit balls are uniformly homeomorphic). In [BL00, Chapter 9], it is shown that if X and Y are ball equivalent, then $X \oplus \mathbb{R} \sim_S Y \oplus \mathbb{R}$ (the other direction: "$X \sim_S Y$ implies that X and Y are ball equivalent" is easy). Therefore, if we consider Banach spectral gap, then there is no serious difference between the sphere equivalence and the ball equivalence.

3. Representation theoretic constants for Cayley graphs

We first give the proof of Theorem A for *Cayley* graph of (finite) groups, and explain where group theory can contribute to this problem. In this section, let G be a finite group, $S \not\ni e$ be a symmetric (finite) generating set of G. Recall the definition of Cayley graphs. The *Cayley graph* of (G, S), written as $\mathrm{Cay}(G, S)$, is constructed as

- the vertex set $V = G$;
- and the edge set $E = \{(g, sg) : g \in G, s \in S\}$.

Example 3.1. $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{\pm 1\})$, $n \geq 3$, is the cycle of lenth n. Although we do not treat in this report, Cayley graphs are also defined for G infinite. In that case, $\mathrm{Cay}(\mathbb{Z}^2, \{\pm(1,0), \pm(0,1)\})$ is the $\mathbb{Z}^2$-lattice in $\mathbb{R}^2$. For a free group F_2 with 2 free generators a, b, $\mathrm{Cay}(F_2, \{a^{\pm 1}, b^{\pm 1}\})$ is the 4-regular tree.

Remark 3.2. Recall that a group G has two natural action on itself: the left multiplication and the right one. We have employed the left multiplication to connect edges in $\mathrm{Cay}(G, S)$, and the right one is left. In fact, this right multiplication becomes a graph automorphsim (in other words, for every $g \in G$, $(v, w) \in E$ iff $(vg, wg) \in E$). Since this right action of G on itself is transitive, $\mathrm{Cay}(G, S)$ is a *vertex-transitive* graph (it means that the automorphism group of the graph acts transitively on the vertex set). Hence, (finite) Cayley graphs are special among all (finite) graphs.

Also recall that in our notaion, we allow graphs to have self-loops and multiple edges. However, if we consider only Cayley graphs, then they do not show up.

3.1. isometric linear representations and displacement constant.

Definition 3.3. We take (G, S) and (X, p).

(1) Define $\pi_{G;X,p} = \pi_{X,p}$ as the left-regular representation of G on $\ell_p(G, \tilde{X}_{(p)})$, namely, for $g \in G$ and $\xi \in \ell_p(G, \tilde{X}_{(p)})$, $\pi_{X,p}(g)\xi(v) := \xi(g^{-1}v)$. Then $\ell_p(G, \tilde{X}_{(p)})$ decomposes as G-representation spaces: $\ell_p(G, \tilde{X}_{(p)}) = \ell_p(G, \tilde{X}_{(p)})^{\pi_{X,p}(G)} \oplus \ell_{p,0}(G, \tilde{X}_{(p)})$. Here the first space is the space of $\pi_{X,p}(G)$-invariant vectors (which consists of "constant functions" form G to $\tilde{X}_{(p)}$); and the second space is the space of "zero-sum" functions, namely,
$$\ell_{p,0}(G, \tilde{X}_{(p)}) := \{\xi \in \ell_p(G, \tilde{X}_{(p)}) : \sum_{v \in G} \xi(v) = 0\}.$$
We omit writting G in $\pi_{G;X,p}$ if G is fixed. We use the same symbol $\pi_{X,p}$ for the restricted representation on $\ell_{p,0}(G, \tilde{X}_{(p)})$.

(2) (p-displacement constant) The *p-displacement constant* of (G, S) on X, written as $\kappa_{X,p}(G, S)$, is defined as
$$\kappa_{X,p}(G, S) := \inf_{0 \neq \xi \in \ell_{p,0}(G, \tilde{X}_{(p)})} \sup_{s \in S} \frac{\|\pi_{X,p}(s)\xi - \xi\|}{\|\xi\|}.$$

Remark 3.4. We will use in the proof of Proposition 5.1 the following norm inequality: for $\xi \in \ell_{p,0}(G, \tilde{X}_{(p)})$, we have

$$\mathrm{dist}(\xi, \ell_p(G, \tilde{X}_{(p)})^{\pi_{X,p}(G)}) \geq \frac{1}{2}\|\xi\|.$$

Indeed, set $\eta = \eta_1 + \eta_0$ for any $\eta \in \ell_p(G, \tilde{X}_{(p)})$ according to the decomposition $\ell_p(G, \tilde{X}_{(p)}) = \ell_p(G, \tilde{X}_{(p)})^{\pi_{X,p}(G)} \oplus \ell_{p,0}(G, \tilde{X}_{(p)})$. Then the map $\eta \mapsto \eta_1$ is given by taking the mean of η. Because the p-mean of the norm is at least the norm of the mean, we have that $\|\eta\| \geq \|\eta_1\|$. Hence for any $\zeta \in \ell_p(G, \tilde{X}_{(p)})^{\pi_{X,p}(G)}$, we have that $\|\xi - \zeta\| \geq \|\zeta\|$ (set $\eta := \xi - \zeta$). Therefore

$$2 \cdot \inf_{\zeta \in \ell_p(G, \tilde{X}_{(p)})^{\pi_{X,p}(G)}} \|\xi - \zeta\| \geq \inf_{\zeta \in \ell_p(G, \tilde{X}_{(p)})^{\pi_{X,p}(G)}} (\|\xi - \zeta\| + \|\zeta\|) \geq \|\xi\|,$$

and we are done.

3.2. **Fundamental lemma for Banach spectral gaps of Cayley graphs.** The following lemma plays a fundamental rôle, which relates p-displacement constant on X to (X, p)-spectral gap for a Cayley graph.

Lemma 3.5. *For a Cayley graph* $\Gamma = \mathrm{Cay}(G, S)$ *and a pair* (X, p)*, we have that*

$$\kappa_{X,p}(G, S)^p \leq \lambda_1(\Gamma; X, p) \leq \frac{|S|}{2} \kappa_{X,p}(G, S)^p.$$

Proof. First note that by Lemma 1.3, $\lambda_1(\Gamma; X, p) = \lambda_1(\Gamma; \tilde{X}_{(p)}, p)$. Take a nonconstant map $f\colon V \to \tilde{X}_{(p)}$ and by replacing f with $f - m(f)$ we may assume $m(f) = 0$. Then we may regard f as a nonzero vector $\xi \in \ell_{p,0}(G, \tilde{X}_{(p)})$. Therefore

$$\begin{aligned}
\lambda_1(\Gamma; X, p) &= \frac{1}{2} \inf_{0 \neq \xi \in \ell_{p,0}(G, \tilde{X}_{(p)})} \frac{\sum_{v \in G} \sum_{s^{-1} \in S} \|\pi_{X,p}(s)\xi(v) - \xi(v)\|^p_{\tilde{X}_{(p)}}}{\|\xi\|^p} \\
&= \frac{1}{2} \inf_{0 \neq \xi \in \ell_{p,0}(G, \tilde{X}_{(p)})} \sum_{s \in S} \left(\frac{\|\pi_{X,p}(s)\xi - \xi\|}{\|\xi\|} \right)^p.
\end{aligned}$$

This ends our proof (note that $\|\pi_{X,p}(s)\xi - \xi\| = \|\pi_{X,p}(s^{-1})\xi - \xi\|$ because $\pi_{X,p}(s)$ is an isometric operator). ☐

Remark 3.6. If we consider $\{(G_n, S_n)\}$ where $\sup_n |S_n| < \infty$, then Lemma 3.5 gives the optimal order estimate between $\kappa_{X,p}(G_n, S_n)$ and $\lambda_1(\mathrm{Cay}(G_n, S_n); X, p)$. However if $\sup_n |S_n| = \infty$, then Lemma 3.5 may not give the precise order.

Nevertheless, if S_n's have "high symmetry", then we have more accurate inequalities. For more precise meaning, we refer the reader to [Mim14, Theorem 3.4], which is based on the work of Pak and Żuk [PZ02].

4. Key propositions on sphere equivalence

4.1. upper moduli and $\mathrm{Sym}(F)$ equivariant homeomorphisms.

Definition 4.1. Let $X \sim_S Y$, and $\phi\colon S(X) \to S(Y)$ be a uniformly continuous map.

(*i*) Define $\mathcal{M}_\phi$ to be the class of all functions $\delta\colon [0, 2] \to \mathbb{R}_{\geq 0}$ which satisfy the following three conditions:
   - δ is nondecreasing;
   - $\lim_{\epsilon \to +0} \delta(\epsilon) = 0$;

- and, for any $x_1, x_2 \in S(X)$ with $\|x_1 - x_2\|_X \leq \epsilon$, we have $\|\phi(x_1) - \phi(x_2)\|_Y \leq \delta(\epsilon)$.

We call an element δ in $\mathcal{M}_\phi$ an *upper modulus of continuity* of ϕ.

(ii) Define $\overline{\phi}\colon X \to Y$ to be the extension of ϕ by homogeneity, namely, $\overline{\phi}(x) := \|x\|_X \phi(x/\|x\|_X)$ for $0 \neq x \in X$ and $\overline{\phi}(0) := 0$. We call $\overline{\phi}$ the *natural extension* of ϕ.

Note that $\overline{\phi}$ is uniformly continuous if we restrict it on a bounded set of X; but that itself is in general *not*.

Example 4.2. In Example 2.2, we have seen the definition of the Mazur map $M_{p,2}\colon S(\ell_p) \to S(\ell_2)$. This map (and also the inverse map) is known to be uniformly continuous, more precisely,

- If $p \geq 2$, then the function $\delta\colon [0,2] \to \mathbb{R}_{\geq 0}$; $\delta(\epsilon) := (p/2)\delta$ is in $\mathcal{M}_{M_{p,2}}$ ($M_{p,2}$ is Lipschitz).
- If $p < 2$, then the function $\delta\colon [0,2] \to \mathbb{R}_{\geq 0}$; $\delta(\epsilon) := 4\delta^{p/2}$ is in $\mathcal{M}_{M_{p,2}}$ ($M_{p,2}$ is $p/2$-Hölder).

Surprislingly, these estimations of Hölder exponents remain to be optimal even when we consider the "noncommutative Mazur map" from noncommutative L_p spaces associated with any von Neumann algebra. This assertion has been recently showed by Ricard [Ric14].

Definition 4.3. Let F be an at most countable set. For a map $\phi\colon S(\ell_p(F,X)) \to S(\ell_q(F,Y))$, we say that ϕ is $\mathrm{Sym}(F)$-*equivariant* if for any $\sigma \in \mathrm{Sym}(F)$, $\phi \circ \sigma_{X,p} = \sigma_{Y,q} \circ \phi$ holds true. Here a Banach space Z and $r \in [1,\infty)$, the symbol $\sigma_{Z,r}$ denotes the isometry $\sigma_{Z,r}$ on $\ell_r(F,Z)$ induced by σ, namely, $(\sigma_{Z,r}\xi)(a) := \xi(\sigma^{-1}(a))$ for $\xi \in \ell_r(F,Z)$ and $a \in F$. Here by $\mathrm{Sym}(F)$ we mean the group of all permutations on F, *including ones of infinite supports.*

For instance, if we consider the Mazur map $M_{p,2}$ as a map from $\ell_p(\mathbb{N},\mathbb{R})$ to $\ell_2(\mathbb{N},\mathbb{R})$, then $M_{p,2}$ is $\mathrm{Sym}(\mathbb{N})$-equivariant. This is because $M_{p,2}$ is coordinatewise.

4.2. **Key proposition for Theorem A.**

Proposition 4.4. *Assune that $\phi\colon S(X) \to S(Y)$ is a uniformly continuous map for two Banach spaces X and Y. Then for any $p \in [1,\infty)$, the map*

$$\Phi = \Phi_p\colon S(\tilde{X}_{(p)}) \to S(\tilde{Y}_{(p)}); \quad (x_i)_i \mapsto (\overline{\phi}(x_i))_i$$

is again a uniformly continuous map that is $\mathrm{Sym}(\mathbb{N})$*-equivariant. Here $\overline{\phi}$ is the natural extension of ϕ and we see $\tilde{X}_{(p)}$ and $\tilde{Y}_{(p)}$, respectively, as $\ell_p(\mathbb{N},X)$ and $\ell_p(\mathbb{N},Y)$.*

Furthermore, if ϕ is α-Hölder, then so is Φ_p. More precisely. if $\delta(t) := Ct^\alpha \in \mathcal{M}_\phi$ for some $C > 0$ and some $\alpha \in (0,1]$, then $\delta'(t) := (2C+2)t^\alpha$ belongs to $\mathcal{M}_{\Phi_p}$.

Proof. By construction, this Φ_p is coordinatewise and hence in particular $\mathrm{Sym}(\mathbb{N})$-equivariant. Our proof of the uniform continuity of Φ_p consists of two cases. Here we only prove the case where $Ct^\alpha \in \mathcal{M}_\phi$ (for general case, we may need to replace δ with larger upper modulus).

Case 1 : *for* $p = 1$. Let $(x_i)_i$ and $(y_i)_i$ be in $S(\tilde{X}_{(1)})$.

First we consider the case where for all $i \in \mathbb{N}$ $\|x_i\|_X = \|y_i\|_X$. Set $r_i := \|x_i\|_X$ and $\epsilon_i r_i = \|x_i - y_i\|_X$. Observe that δ is concave in $[0, 2]$. By the Jensen inequality, we have the following:

$$\|\Phi((x_i)_i) - \Phi((y_i)_i)\|_{\tilde{Y}_{(1)}} \leq \sum_i r_i \delta(\epsilon_i) \leq \delta(\sum_i r_i \epsilon_i) = \delta(\|(x_i)_i - (y_i)_i\|_{\tilde{X}_{(1)}}).$$

Secondly we deal with the general case. For $(x_i)_i, (y_i)_i \in \tilde{X}_{(1)}$, define $z_i := \frac{\|x_i\|_X}{\|y_i\|_X} y_i$ ($z_i := x_i$ if $y_i = 0$). Suppose $\|(x_i)_i - (y_i)_i\|_{\tilde{X}_{(1)}} \leq \epsilon$. Because for any i, $\|x_i - y_i\|_X \geq \|z_i - y_i\|_X$, we have that $\|(z_i)_i - (y_i)_i\|_{\tilde{X}_{(1)}} \leq \epsilon$. Hence we obtain that $\|(x_i)_i - (z_i)_i\|_{\tilde{X}_{(1)}} \leq 2\epsilon$. Therefore in the first argument, we have that $\|\Phi((x_i)_i) - \Phi((z_i)_i)\|_{\tilde{Y}_{(1)}} \leq \delta(2\epsilon)$. Since $\|\Phi((y_i)_i) - \Phi((z_i)_i)\|_{\tilde{Y}_{(1)}} \leq \epsilon$ by homogeneity, we conclude that $\delta'(t) := \delta(2t) + t = 2^\alpha C t^\alpha + t$ belongs to $\mathcal{M}_{\Phi_1}$.

Case 2 : for general $p > 1$. First observe that $t \in [0, 2^{1/p}]$, we have that $\delta(t)^p \leq C^{p-1}\delta(t^p)$. Then the remaining argument goes along a similar line to one in Case 1. Thus we can show that $\delta'(t) := (C^{p-1}\delta((2t)^p))^{1/p} + t = 2^\alpha C t^\alpha + t$ belongs to $\mathcal{M}_{\Phi_p}$.

In each case, finally observe that for $t \in [0, 2]$, $(2C + 2)t^\alpha \geq 2^\alpha C t^\alpha + t$. □

Lemma 9.9 in [BL00] showed the first assertion above. However, the estimation of upper moduli is worse than in this proposition, and did not verify the latter assertions.

4.3. **Generalized Mazur map: key proposition for Theorem B.**

Theorem 4.5. *For any uniformly convex Banach space X and $p, q \in (1, \infty)$, we have that $\tilde{X}_{(p)} \sim_S \tilde{X}_{(q)}$. Furthermore, we may have a uniform homeomorphism $\phi\colon S(\ell_p(\mathbb{N}, X)) \to S(\ell_q(\mathbb{N}, X))$ which is* $\mathrm{Sym}(\mathbb{N})$*-equivariant.*

Proof. Choose $1 < p_0 < \min\{p, q\}$ and $\infty > p_1 > \max\{p, q\}$. Then [BL76, Theorem 5.1.2] applies to the case where $\Omega = \mathbb{N}$ and $A_0 = A_1 = X$. This tells us that both of $\tilde{X}_{(p)}$ and $\tilde{X}_{(q)}$ are, respectively, isometrically isomorphic to some intermediate points of a complex interpolation pair $(\tilde{X}_{(p_0)}, \tilde{X}_{(p_1)})$. Because $\tilde{X}_{(p_0)}$ and $\tilde{X}_{(p_1)}$ are uniformly convex, the result mentioned in Example 2.3 applies.

The last assertion follows from the proof of [BL00, Theorem 9.12]. Indeed, the definition of f_x for $x \in \ell_p(\mathbb{N}, X)$, as the minimizer of a certain norm, in Proposition I.3 in [BL00] is $\mathrm{Sym}(\mathbb{N})$-equivariant in the current setting. □

This map may be regareded as a *generalized Mazur map* because it coincide with the usual Mazur map if we consider the complex interpolation pair (ℓ_{p_0}, ℓ_{p_1}) in the proof (for $X = \mathbb{R}$). However, note that we are only able to define it for $p, q > 1$, as long as we employ the complex interpolation.

5. Proof of Theorem A for Cayley graphs

This part is based on a work of Bader–Furman–Gelander–Monod [BFGM07]. See Section 4.a in [BFGM07] for the original idea of them. We will show the following proposition concerning the p-displacement constants.

Proposition 5.1. *Let $X \sim_S Y$ and $\phi\colon S(X) \to S(Y)$ be a uniform homeomorphism. Let G be a finite group, $S \not\ni e$ be a symmetric (finite) subset. Then for any $p \in [1,\infty)$, we have the following inequality:*

$$\kappa_{X,p}(G,S) \geq \delta_1^{-1}\left(\frac{1}{2}\delta_2^{-1}\left(\frac{1}{2}\right)\kappa_{Y,p}(G,S)\right).$$

Here $\delta_1 \in \mathcal{M}_{\Phi_p}$ and $\delta_2 \in \mathcal{M}_{\Phi_p^{-1}}$.

Proof. By Proposition 4.4, $\Phi_p\colon \tilde{X}_{(p)} \to \tilde{Y}_{(p)}$ is a uniform homeomorphism that is $\mathrm{Sym}(\mathbb{N})$-equivariant. By coordinate transformation, we may regard Φ_p as

$$\Phi_p\colon S(\ell_p(G,\tilde{X}_{(p)})) \to S(\ell_p(G,\tilde{Y}_{(p)}))$$

(note that $\ell_p(G,\tilde{X}_{(p)}) \simeq \tilde{X}_{(p)}$), which is $\mathrm{Sym}(G)$-equivariant. We thus have that $\Phi_p \circ \pi_{X,p} = \pi_{Y,p} \circ \Phi_p$. Note that we consider $\pi_{X,p}$ and $\pi_{Y,p}$ as G-representations, respectively, on $\ell_p(G,\tilde{X}_p)$ and $\ell_p(G,\tilde{Y}_p)$, *not* on $\ell_{p,0}$.

Choose any $\xi \in S(\ell_{p,0}(G,\tilde{X}_{(p)})) \subseteq S(\ell_p(G,\tilde{X}_{(p)}))$ and set $\eta := \Phi_p(\xi) \in S(\ell_p(G,\tilde{Y}_{(p)}))$. We warn that η does *not* belong to $S(\ell_{p,0}(G,\tilde{Y}_{(p)}))$ in general. We however overcome this difficulty in the following argument. Recall that $\ell_p(G,\tilde{X}_{(p)})$ is decomposed as the direct sum of $\ell_p(G,\tilde{X}_{(p)})^{\pi_{X,p}(\Gamma)}$ and $\ell_{p,0}(G,\tilde{X}_{(p)})$. Note that the former subspace is sent to $\ell_p(G,\tilde{Y}_{(p)})^{\pi_{Y,p}(G)}$ by Φ_p (again because Φ_p is $\mathrm{Sym}(G)$-equivariant). Recall the inequality in Remark 3.4 and get that $\mathrm{dist}(\xi,\ell_p(G,\tilde{X}_{(p)})^{\pi_{X,p}(G)}) \geq \frac{1}{2}$.

In particular, from this we have that $\mathrm{dist}(\xi, S(\ell_p(G,\tilde{X}_{(p)})^{\pi_{X,p}(\Gamma)})) \geq \frac{1}{2}$. Therefore, by the uniform continuity of Φ_p^{-1}, we have that $\mathrm{dist}(\eta, S(\ell_p(G,\tilde{Y}_{(p)})^{\pi_{Y,p}(\Gamma)})) \geq \delta_2^{-1}\left(\frac{1}{2}\right)$.

Decompose η as $\eta = \eta_1 + \eta_0$ where $\eta_1 \in \ell_p(G,\tilde{Y}_{(p)})^{\pi_{Y,p}(G)}$ and $\eta_0 \in \ell_{p,0}(G,\tilde{Y}_{(p)})$. We claim that

$$\|\eta_0\| \geq \frac{1}{2}\delta_2^{-1}\left(\frac{1}{2}\right).$$

Indeed, let $\eta_1' := \eta_1'/\|\eta_1'\|$ (if $\eta_1 = 0$, then set η_1' as any vector in $S(\ell_p(G,\tilde{Y}_{(p)})^{\pi_{Y,p}(G)})$). Then by the inequality in the paragraph above, we have that $\|\eta - \eta_1'\| \geq \delta_2^{-1}\left(\frac{1}{2}\right)$. Because $\|\eta_1\| \geq 1 - \|\eta_0\|$, we also have that $\|\eta_1 - \eta_1'\| \leq \|\eta_0\|$ and that $\|\eta - \eta_1'\| \leq \|\eta - \eta_1\| + \|\eta_1 - \eta_1'\| \leq 2\|\eta_0\|$. By combining these inequalities, we prove the claim.

By the definition of $\kappa_{Y,p}(G,S)$, we have that

$$\sup_{s\in S}\|\pi_{Y,p}(s)\eta - \eta\| = \sup_{s\in S}\|\pi_{Y,p}(s)\eta_0 - \eta_0\| \geq \|\eta_0\|\kappa_{Y,p}(G,S) \geq \frac{1}{2}\delta_2^{-1}\left(\frac{1}{2}\right)\kappa_{Y,p}(G,S).$$

Finally, because $\Phi_p \circ \pi_{X,p} = \pi_{Y,p} \circ \Phi_p$, we conclude by the uniform continuity of Φ_p that

$$\sup_{s\in S}\|\pi_{X,p}(s)\xi - \xi\| \geq \delta_1^{-1}\left(\frac{1}{2}\delta_2^{-1}\left(\frac{1}{2}\right)\kappa_{Y,p}(G,S)\right).$$

By taking the infimum over $\xi \in S(\ell_{p,0}(G,\tilde{X}_{(p)}))$, we obtain the desired assertion. □

By combining the proposition above, Proposition 4.4, and Lemma 3.5, we obtain the conclusion in Theorem A for Γ a Cayley graph.

6. The Gross trick

In this section, we give the proof of Theorem A for Γ arbitrary finite graph. To do this, our idea is to consider *Schreier coset graphs* and to reduce all cases to these ones. The Gross theorem, which we will mention later, enables us to perform the latter procedure. The author call this trick the *Gross trick*.

6.1. **Schreier coset graph.** In the proof of Lemma 3.5 and Proposition 5.1, it may be noticed that we have *never* employed the right regular representation. This means, we only need group multiplication only on one side, which was used to connect the edges. From this observation, we encounter with the conception of *Schreier coset grpahs.*

Definition 6.1. Let G be a finitely generated group, S be a symmetric finite generating set, and H be a subgroup of G *of finite index*. By $\mathrm{Sch}(G, H, S)$ we mean the *Schreier coset graph*, that is

- the vertex set is the left cosets: $V = G/H$;
- the edge set $E := \{(gH, sgH) : gH \in G/H, s \in S\}$.

Remark 6.2. One remark is that we may take G as a finite group in the definition.

The other remark is that, unlike Cayley grpahs, Schreier coset graphs in general have *no* symmetry at all (note that only possible muliplication on G/H is from the left, but this is used for connecting edges). Moreover, in general $\mathrm{Sch}(G, H, S)$ may have self-loops and multiple edges.

Once we employ the concept of Schreier coset graphs, we have a similar definition of *p-displacement constants* for the triple (G, H, S) in terms of the quasi-regular representation of G on $\ell_{p,0}(G/H, \tilde{X}_{(p)})$. Furthermore, we have exactly the same inequalities as ones in Lemma 3.5 and Proposition 5.1 for Schreier coset graphs. In this report, we omit the precise forms. Instead, we refer the reader to Definition 3.1, Lemma 3.3, and Proposition 4.2 in [Mim14].

Thus we ends the proof of Theorem A for the case where Γ is a Shreier coset graph.

6.2. **the Gross trick.** Now we explain the main trick on the proof. This employs the following result of Gross.

Theorem 6.3 ([Gro77])**.** *Any finite connected and* regular *graph* (*possibly with multiple edges and self-loops*) *with* even *degree can be realized as a Schreier coset graph.*

Remark 6.4. The proof of Gross's theorem is based on the "2-factorization" of such a graph (Petersen). This means, for such a graph, we can decompose the (undirected) edge set as the disjoint union of 2-regular graphs (cycles). From these cycles, we can endow Γ with the structure of a Schreier coset graph. Hence this realization is not just the existence, but not sufficiently concrete or handlable in general setting.

Also, by passing to apropriate limits, the Gross theorem can be extended to infinite regular conncected graphs of even degree.

This theorem of Gross roughly asserts that Schreier coset graphs are "more or less universal" among graphs of uniformly bounded degree (compare with speciality of Cayley graphs!). More precise meaing of "universal" will be explained in the usage of "Gross trick", as below.

The following argument is the *Gross trick*: Let $\Gamma = (V, E)$ be a finite connected graph. Then we take the *even regularization* of Γ in the following sense: we let V unchanged. We first double each edge in E. Note that then for any $v, w \in V$, $\deg(v) - \deg(w) \in 2\mathbb{Z}$ and that the maximum degree is $2\Delta(\Gamma)$. Finally, we let a vertex v whose degree is $2\Delta(\Gamma)$ unchanged, and for all the other vertices add, respectively, appropriate numbers of self-loops to have the resulting degree $= 2\Delta(\Gamma)$ for each vertex. We write the resulting graph as $\Gamma' = (V, E')$. Then by the Gross theorem, Γ' can be realized as a Schreier coset graph and thus the argument in Subsection 6.1 applies to Γ'. Finally observe that $\lambda_1(\Gamma'; Z, p) = 2\lambda_1(\Gamma; Z, p)$ for any Banach space Z because self-loops do not affect the spectral gap.

This completes our proof of Theorem A for general graphs Γ.

7. Proofs of Theorem B and Corollary C

Proof of Theorem B. Let $X \sim_S Y$, where Y is uniformly convex and let $p, q \in (1, \infty)$. By Theorem 4.5, there exists an $\mathrm{Sym}(\mathbb{N})$-equivariant uniform homeomorphism $\Phi := \Phi_{p,q} \colon S(\tilde{Y}_{(p)}) \to S(\tilde{Y}_{(q)})$. First we start from the case where Γ is of the form $\mathrm{Sch}(G, H, S)$. Then we regard Φ as an $\mathrm{Sym}(G/H)$-equivariant uniform homeomorphism

$$\Phi \colon S(\ell_p(G/H, \tilde{Y}_{(p)})) \to S(\ell_q(G/H, \tilde{Y}_{(q)})).$$

We thus may apply a similar argument to Proposition 5.1 to the pair $((Y, p); (Y, q))$. Because Proposition 5.1 works for the pairs $((X, p); (Y, p))$ and $((Y, q); (X, q))$, we are done.

For general cases, apply the Gross trick. $\square$

Proof of Corollary C. The first assertion holds true by Theorem A, Theorem B, and the fact of that uniformly curved Banach spaces are isomorphic (and in particular sphere equivalent) to some uniformly convex Banach spaces, see Remark 1.19. The second assertion holds true for the following reason: if $X \in [\ell_2]_S$, then by Theorem A and Lemma 1.3, the (X, p)-ander property is equivalent to the $(\mathbb{R}, p)$-ander property. The original Matoušek extrapolation enables us to extend our results even for $p = 1$. $\square$

8. Application: embeddings of Hamming graphs into noncommutative L_p spaces

As an application of our main results, we consider embeddings of Hamming graphs into noncommutative L_p spaces associated with arbitrary von Neumann algebras. For $d \geq 1$ and $k \geq 2$, the *Hamming graph* $H(d, k)$ is defined as the following:

- the vertex set V is the set of the ordered d-tuples of T, $|T| = k$;
- the edge set E consists of all pairs which diffres in precisely one coordinate.

In other words, $H(d, k)$ is the product of d copies of the complete graph K_k on k vertices. It is easy to see that $H(d, k)$ is $d(k-1)$-regular and $\mathrm{diam}(H(d, k)) = d$. As a byproduct of Theorem A, we have the following:

Theorem 8.1. *Let $\mathcal{M}$ be a von Neumann algebra. By $L_p(\mathcal{M})$, we denote the noncommutative L_p space associated with $\mathcal{M}$.*

(1) *For $p \in [1, 2)$, then we have that $\lambda_1(H(d, k); L_p(\mathcal{M}), p) \asymp_p k$.*
(2) *For $p \in [2, \infty)$, then we have that $\lambda_1(H(d, k); L_p(\mathcal{M}), 2) \asymp_p k$.*

Note that the multiplicative constants in these estimation do *not* depend on d, k, and $\mathcal{M}$; and only depend on p.

Proof. We only prove the case where k is a prime number. For other cases, we use a similar technique to the Gross trick (namely, we add multiple edges and self-loops to have better graph) in order to apply [Mim14, Theorem 3.4].

Note that $H(d,k) = \mathrm{Cay}(G_{d,k}, S_{d,k})$, where $G_{d,k} = (\mathbb{Z}_k)^d$ and $S_{d,k}$ consists of vectors whose exactly one coordinate is non-zero. Then we can apply [Mim14, Theorem 3.4] (see also Remark 3.6) with $\nu = 1$ and we have that

$$\lambda_1(H(d,k); X, q) = \frac{d(k-1)}{2}\kappa_{X,q}(G_{d,k}, S_{d,k})^q.$$

Recall that by the result of Ricard [Ric14] (see also Example 4.2) the noncommutative Mazur map, which we also write $M_{p,2}$, is

- $p/2$-Hölder if $p \in [1,2]$;
- and Lipschitz if $p > 2$.

(Note that multiplicative constants do not depend on $\mathcal{M}$ in direct sumargument.) By spectral calculus, it is not difficult to show that $\lambda_1(H(d,k); \mathbb{R}, 2) = k$, and so

$$\kappa_{X,2}(G_{d,k}, S_{d,k}) = \left(\frac{2k}{d(k-1)}\right)^{1/2}.$$

Therefore by Proposition 5.1, we have that

- $\lambda_1(H(d,k); L_p(\mathcal{M}), p) \gtrsim_p k$ for $p \in [1,2]$;
- and $\lambda_1(H(d,k); L_p(\mathcal{M}), 2) \gtrsim_p k$ for $p > 2$.

(For the former inequalities, see that $\ell_p(\mathbb{N}, L_p(\mathcal{M}))$ is again a noncommutative L_p space.) Finally, we will prove the converse order inequalities. For $p \in [1,2]$, consider the following mapping

$$f_p \colon H(d,k) \to \ell_p(d, \ell_p(T, \mathbb{R})); \quad (a_1, \dots a_d) \mapsto (\chi_{\{a_1\}}, \dots, \chi_{\{a_d\}}).$$

Here T is the base set ($|T| = k$) of $H(d,k)$, and χ stands for the characteristic function. Then simple calculation shows that

$$\frac{1}{2}\frac{\sum_{v\in V_{d,k}}\sum_{e=(v,w)\in E_{d,k}}\|f_p(w) - f_p(v)\|^p}{\sum_{v\in V_{d,k}}\|f_p(v) - m(f_p)\|^p} = \frac{k^p}{(k-1)^{p-1}+1} \asymp_p k$$

(note that $k \geq 2$). Because $\ell_p(\mathbb{N}, L_p(\mathcal{M}))$ contains ℓ_p, this shows that $\lambda_1(H(d,k); L_p(\mathcal{M}), p) \asymp_p k$ for $p \in [1,2]$. For $p > 2$, because ℓ_2 is an isometric subspace of $L_p((0,1))$, we can approximately embed $H(d,k)$ into $\ell_2(\mathbb{N}, L_p(\mathcal{M}))$ by using f_2 by approximating (finitely many) elements in $L_p((0,1))$ by step functions in ℓ_p. This gives that $\lambda_1(H(d,k); L_p(\mathcal{M}), 2) \lesssim_p k$ and therefore $\lambda_1(H(d,k); L_p(\mathcal{M}), 2) \asymp_p k$. ☐

Corollary 8.2. *In the setting of Theorem 8.1, the following hold true.*

(i) (1) *For $p \in [1,2)$, $c_{L_p(\mathcal{M})}(H(d,k)) \asymp_p d^{1-1/p}$.*
 (2) *For $p \in [2,\infty)$, $c_{L_p(\mathcal{M})}(H(d,k)) \asymp_p d^{1/2}$.*

(ii) *For an infinite sequence $\{H(d_n,k_n)\}_n$ with $\lim_{n\to\infty} d_n = \infty$, the following hold:*
 (1) *For $p \in [1,2)$, the supremum of the exponents $\alpha \in [0,1]$ such that there exists $C > 0$ such that (t^α, Ct) can be a control pair of $\coprod_n H(d_n,k_n)$ into $L_p(\mathcal{M})$ is $1/p$.*
 (2) *For $p \in [2,\infty)$, the supremum of the exponents $\alpha \in [0,1]$ such that there exists $C > 0$ such that (t^α, Ct) can be a control pair of $\coprod_n H(d_n,k_n)$ into $L_p(\mathcal{M})$ is $1/2$.*

Proof. On (i), in both cases, inequalities from below follow from Theorem 1.11 and Theorem 8.1. Inequalities from above can be deduced from the special embeddings of $H(d_n,k_n)$ indicated in the proof of theorem 8.1.

On (ii), inequalities from above follow from the estimations on distoritons in (i) and Lemma 1.13. Ones from below are again from the special embeddings above. □

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