Følner type sets, Property A and coarse embeddings

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Abstract

Our goal is to expose amenability as a tool to produce good embeddings of metric spaces into Banach spaces. After introducing amenability, focussing on Følner's isoperimetric criterion, we show how Yu's property A generalizes the notion to uniformly discrete metric spaces. We show how to produce proper isometric actions of amenable groups and coarse embeddings of metric spaces with Property A. Finally, by keeping track of the size of Følner sets, we obtain lower bounds on the compression functions of those embeddings.

1 Amenability and proper actions on Hilbert spaces

1.1 The Hausdorff-Banach-Tarski paradox and von Neumann's definition

The Hausdorff-Banach-Tarski paradox states that it is possible to cut a sphere into finitely many pieces and reassemble them with no deformations into two spheres of the same size as the original one. It is called a paradox only because it contradicts our geometrical intuition in a very strong sense. What makes such a cutting possible lies in the use of the axiom of choice and of non-Lebesgue-measurable pieces. In the study of that theorem, the notion of amenability arose as a fundamental group theoretic property forbidding such decompositions.

Theorem 1.1 (Hausdorff, 1914 [Hau] - Banach,Tarski, 1924 [BT])**.** *Let* $X=\mathbb{S}^2$ *denote the two dimensional unit sphere in* $\mathbb{R}^3$ *and let* $G = \mathrm{SO}_3(\mathbb{R})$ *be its group* *of isometries. There exists a non-measurable partition of X into four subsets $A_1, A_2,$ and B_1, B_2 and rotations $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G$ such that*

$$(\alpha_1 \cdot A_1) \sqcup (\alpha_2 \cdot A_2) = G, \text{ and } (\beta_1 \cdot B_1) \sqcup (\beta_2 \cdot B_2) = G.$$

Proof: Consider the subgroup $F = \mathrm{F}(\alpha, \beta)$ of G generated by the two matrices

$$\alpha = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}$$

and admit that this subgroup is free. Consider the following partition of F into four subsets :

$$\begin{aligned} \mathcal{A}_+ &= \{\text{reduced words starting with the letter } \alpha\} \\ \mathcal{A}_- &= \{\text{reduced words starting with the letter } \alpha^{-1}\} \\ \mathcal{B}_+ &= \{\text{reduced words starting with the letter } \beta\} \cup \{\beta^{-n},\ n \geq 0\} \\ \mathcal{B}_- &= \{\text{reduced words starting with the letter } \beta^{-1}\} \setminus \{\beta^{-n},\ n \geq 0\} \end{aligned}$$

These sets satisfy the following :

$$\mathcal{A}_+ \sqcup \alpha\mathcal{A}_- = G \text{ and } \mathcal{B}_+ \sqcup \beta\mathcal{B}_- = G.$$

Now fix a set of representatives $\{x_i\}_{i\in\mathcal{I}}$ of the F-orbits in X and define

$$\begin{aligned} A_1 &= \{g \cdot x_i,\ g \in \mathcal{A}_+,\ i \in \mathcal{I}\}, & A_2 &= \{g \cdot x_i,\ g \in \mathcal{A}_-,\ i \in \mathcal{I}\}, \\ B_1 &= \{g \cdot x_i,\ g \in \mathcal{B}_+,\ i \in \mathcal{I}\}, & B_2 &= \{g \cdot x_i,\ g \in \mathcal{B}_-,\ i \in \mathcal{I}\}. \end{aligned}$$

We obtain that $X = A_1 \sqcup (\alpha \cdot A_2) = B_1 \sqcup (\beta \cdot B_2)$. $\square$

Such a decomposition is called a *paradoxical decomposition*. From his study of the Banach-Tarski Paradox, Von Neumann came up with the following definition :

Definition 1.2 (von Neumann, 1929 [vN]). Let G be a discrete group, a *mean* on G is a linear functional $M : \ell^\infty(G) \to \mathbb{R}$ which satisfies

1. $M(f) \geq 0$ whenever $f \geq 0$,
2. $M(\mathbf{1}) = 1$.

A mean is called *left-invariant* if additionally

3. $M(g \cdot f) = M(f)$, for every $g \in G,\ f \in \ell^\infty(G)$.

G is called *amenable* if it admits a left-invariant mean.

Remark 1.3. To get a intuitive understanding of the notion, it is important to note that evaluating a left-invariant mean on indicator functions of subsets of G will give us a left invariant *finitely-additive measure* on G.

The crucial observation of von Neumann is that the existence of paradoxical decompositions of the group is an obstruction to amenability. Tarski later proved that it is actually the only obstruction.

Theorem 1.4 (Tarski, 1938 [Ta])**.** *A discrete group G admits a paradoxical decomposition if and only if it is not amenable.*

In a modern view-point, theorem 1.1 uses non-amenability of a certain isometric action of the free group on the sphere to produce a paradoxical decomposition of that sphere. It is difficult to prove amenability or non-amenability of a group using this definition but let's see some examples.

Example 1.5.

1. Every finite group is amenable. Averaging a function amongst the elements of the group provides a left-invariant mean.
2. Free groups are non-amenable. The case of two generators follows from the proof of Theorem 1.1 and the argument for more generators is completely similar.
3. The group $\mathbb{Z}$ of all integers is an amenable group. Providing an explicit left-invariant mean is impossible since it relies on the axiom of choice. One such mean could be given by taking the limit of bounded functions along a $\mathbb{Z}$-invariant ultrafilter.

1.2 Følner's criterion

The most surprising fact about the concept of amenability is that it admits many equivalent definitions coming from very diverse areas of mathematics : measure theoretic, geometric, dynamical, analytic, spectral, etc. The most important for our exposition is the Følner geometric characterization in terms of sets with small boundaries.

Definition 1.6. Let G be a finitely generated group equipped with the word metric associated to some finite generating set, let A be a subset of G, and let $R > 0$. Define the *R-boundary* of A as

$$\partial_R A = \{g \in G \setminus A \mid d(g, A) \leq R\}.$$

Fix $\varepsilon > 0$, a finite subset A of G is called an (R, ε)-*Følner set* if it satisfies

$$\frac{\#\partial_R A}{\#A} \leq \varepsilon$$

This definition is well-suited to give an intuitive notion of Følner sets as sets with small boundaries, however it is almost always more practical to work with the following :

Definition 1.5 (revisited)**.** A finite subset $A \subset G$ is called an (R, ε)-*Følner set* if it satisfies

$$\frac{\#(g \cdot A \triangle A)}{\#A} \leq \varepsilon$$

for every $g \in G$ such that $|g| \leq R$.

The equivalence between the two definitions relies on the fact that the size of the symmetric difference between A and one of its close translates is roughly equal to the size of its boundary. Note that in order to pass from one definition to the other we may have to multiply ε or R by some fixed constant.

Theorem 1.6 (Følner, 1955 [Føl])**.** *A finitely generated group G is amenable if and only if, for every $\varepsilon > 0$ and for every $R > 0$, G contains an (R, ε)-Følner set.*

Remark 1.7. Fixing $R = 1$ in the theorem would give the exact same class of groups. This is due to the fact that R-boundaries for large R can be controlled in terms of 1-boundaries. So to obtain an (R, ε)-Følner set, one can choose a $(1, \delta)$-Følner set for a sufficiently small δ. In this setting, a sequence of $(1, \varepsilon_n)$-Følner sets (F_n) is called a *Følner sequence* if $\varepsilon_n \to 0$. It will always satisfy

$$\lim_{n \to \infty} \frac{\#g \cdot F_n \triangle F_n}{\#F_n} = 0$$

However, it is very convenient to keep the flexibility of fixing R

Proof: We only give a sketch.

Suppose that G satisfies Følner's criterion and let $F_n \subset G$ be $(n, \frac{1}{n})$-Følner sets. Define functionals M_n on $\ell^\infty(G)$ by

$$M_n(\varphi) = \frac{1}{\#F_n} \sum_{g \in F_n} \varphi(g).$$

The M_n are unit functionals on $\ell^\infty(G)$, and by compactness of the unit sphere in $\ell^\infty(G)^*$ we can assume that the sequence (M_n) converges to a weak-* limit

M. It is easy to check that M is a mean, and left-invariance is a consequence of the asymptotic invariance of the F_n's.

For the converse, we use that $\ell^1(G)$ is dense in its bidual $\ell^\infty(G)^*$. Given M a left-invariant mean, choose a sequence $\phi_n \in \ell^1(G)$ of finite support functions converging to M. Moreover, choose each ϕ_n so that there exist $N > 0$ such that ϕ_n takes value in $\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1\}$. By left-invariance of M, $g \cdot \phi_n - \phi_n$ must become small as n goes to infinity. Considering the sets $F_n^k = \{x \in G \mid \phi_n(x) \leq \frac{k}{N}\}$, we see that by a pigeon-hole principle, at least one of them must be close to its translate by g. $\square$

Let us now revisit our previous examples from Følner's point of view.

Example 1.8.

1. Every finite group is amenable. Indeed, the group itself is an (R, ε)-Følner set for any R and ε.

2. Free groups are non-amenable. Indeed, the Cayley graph of a free group of rank k is a $2k$-regular tree. We can easily check that any connected sub-tree containing n points has a 1-boundary of size $n(2k-2)+2$ forbidding the existence of $(1, \varepsilon)$-Følner sets for small values of ε.

3. The group $\mathbb{Z}$ of all integers is an amenable group. Intervals of the form $[0, n]$ are (R, ε)-Følner at least when $n > \varepsilon/R$.

4. One goes easily from $\mathbb{Z}$ to $\mathbb{Z}^d$ and to any abelian finitely generated group.

Følner's criterion naturally raises the following question : when does an infinite sequence of balls form a Følner sequence? The following gives a complete answer to this question.

Corollary 1.9. *All groups with subexponential growth are amenable.*

Proof: We'll prove the converse statement, i.e. that non-amenabe groups have exponential growth.

Let G be a finitely generated group. Denote by $B(n)$ the ball of radius n and by $S(n)$ the sphere of radius n in G. We have

$$\begin{aligned}
\#B(n) &= \#B(n-1) + \#S(n) \\
&= \#B(n-1)\left(1 + \frac{\#S(n)}{\#B(n-1)}\right) \\
&= \#B(n-2)\left(1 + \frac{\#S(n-1)}{\#B(n-2)}\right)\left(1 + \frac{\#S(n)}{\#B(n-1)}\right) \\
&= \prod_{i=1}^{n}\left(1 + \frac{\#S(i)}{\#B(i-1)}\right).
\end{aligned}$$

It is immediate that $\partial B(n) = S(n+1)$, so by non-amenability of G, the general term of the product must be uniformly bounded away from 1. This implies exponential growth. $\square$

Note that the proof also tells us that in a non-amenable group of sub-exponential growth, at least a subsequence of the balls forms a Følner sequence.

1.3 Gromov's a-T-menability

Let us recall a few facts about groups actions on Hilbert spaces.

Definition 1.10. An *affine isometric action* α of G on a Banach space E is a homomorphism of G into the group of affine isometric transformations $\mathrm{Aff}(E)$.

Such an action is called *proper* if moreover for some (equivalently for all) $\xi \in E$

$$\|\alpha(g)\xi\| \to \infty \text{ whenever } |g| \to \infty$$

Definition 1.11 (Gromov, 1988 [**?**])**.** A group G is called *a-T-menable* if it admits a proper affine isometric action on a Hilbert space.

A-T-menability was introduced by Gromov as a strong negation of Kazhdan's property (T) which requires that every affine isometric action of the group on a Hilbert space has bounded orbits. The terminology follows from the fact that a-T-menability is a weak form of amenability, although this is not clear from the definition.

Example 1.12.

1. $\mathbb{Z}^d$ is a-T-menable. Indeed, the action

   $$\alpha(m_1, \ldots, m_d)(x_1, \ldots, x_d) = (x_1 + m_1, \ldots, x_d + m_d)$$

   is proper.

2. The free group on two generators $\mathbb{F}_2 = F(a,b)$ acts properly on a Hilbert space.

   Proof: Consider the action of $\mathbb{F}_2$ on its Cayley graph $\Gamma = (V, E)$ for the standard generating set. Equip Γ with the natural orientation where edges have positive orientation from g to ag or bg and negative orientation otherwise. Consider now the Hilbert space $\mathcal{H} = \ell^2(E)$ of square summable functions on the edges of Γ. The left-action of G on Γ lifts to a unitary representation of $\mathcal{H}$. Define now $b : G \to \mathcal{H}$ by

   $$b(g)(e) = \begin{cases} 1 & \text{if } e \notin [e, g] \\ -1 & \text{if } e \notin [g, e] \\ 0 & \text{otherwise} \end{cases}$$

where $[x, y]$ denotes the oriented geodesic from x to y. It is easily checked that the formula

$$\alpha(g)\xi = g \cdot \xi + b(g)$$

defines a proper affine isometric action of G. $\square$

The following theorem shows that amenable groups are a-T-menable, it is essential to us since it gives an explicit construction of a proper action given Følner sets on the group. The same approach will be applied in the non-equivariant setting and in both cases we will be able to obtain quantitative information about the actions (resp. coarse maps) obtained this way.

Theorem 1.13 (Bekka-Cherix-Valette, 1993 [BCV]). *Any amenable group admits a proper affine isometric action on a Hilbert space.*

Proof: Let G be an amenable group, and let F_n be $(n, 1/n^2)$-Følner sets in G. Consider the Hilbert sum $\mathcal{H} = \bigoplus_{i=1}^{\infty} \ell^2(G)$ equipped with the natural diagonal action of G. Now define $\xi_n \in \ell^2(G)$ by

$$\xi_n = \frac{1}{\sqrt{(\#F_n)}} \chi_{F_n},$$

where χ_{F_n} denotes the indicator function of F_n, and define $b(g) \in \mathcal{H}$ by $b(g) = \bigoplus_n g \cdot (\xi_n - \xi_n)$. Note that $b(g)$ belongs to $\mathcal{H}$ since

$$\begin{aligned} \|b(g)\|^2 &= \sum_{n=1}^{\infty} \|g \cdot \xi_n - \xi_n\|^2 \\ &= \sum_{n=1}^{\infty} \frac{\#(g \cdot F_n \triangle F_n)}{\#F_n} \end{aligned}$$

and by Følner's condition when n becomes large enough, the summand is dominated by $1/n^2$ which insures that the series converges. Define $\alpha : G \to \mathrm{Aff}(\mathcal{H})$ by

$$\alpha(g)v = g \cdot v + b(g).$$

This is a well-defined affine isometric action of G. To see that it is proper, notice that as $|g|$ grows larger and larger, so does the amount of indices n sucht that $g \cdot F_n$ and F_n are disjoint. Hence

$$\begin{aligned} \|b(g)\| &\geq 2 \cdot \#\{n \mid F_n \cap g \cdot F_n = \emptyset\} \\ &\to \infty \quad \text{as } |g| \to \infty. \end{aligned}$$

$\square$

2 Property A and coarse embeddings

2.1 Property A

Definition 2.1 (Yu, 2000 [Yu]). Let X be a uniformly discrete metric space. We say that X has *Property A* if for every $\varepsilon > 0$ and $R > 0$, there exists a collection $(A_x)_{x\in X}$ of finite subsets of $X \times \mathbb{N}$ and $S > 0$ such that

(a) $\dfrac{\#A_x \triangle A_y}{\#A_x \cap A_y} \leq \varepsilon$ whenever $d(x, y) \leq R$, and

(b) $A_x \subset B(x, S) \times \mathbb{N}$.

Such subsets are called (R, ε)-Følner type sets.

Observe that condition (a) is similar to Følner's condition; sets associated to close points are close. Condition (b), however, replaces equivariance. Indeed, in group it is always the case that finite subsets are disjoint from their far translates. Here, we make it a requirement.

The use of the extra dimension $\mathbb{N}$ allows us to count points with multiplicity and is necessary for technical reasons.

Example 2.2. Amenable groups, seen as uniformly discrete spaces have property A. Indeed, fix $R, \varepsilon > 0$ and let F be a (R, δ)-Følner set for a suitable δ. Then the family of sets $A_g = gF \times \{1\}$ satisfies property A for R and ε.

The question whether Property A for groups is equivalent to amenability is natural and the following example shows that it isn't. Indeed, free groups have trees as Cayley graphs.

Example 2.3. Infinite trees have property A.

Proof: Let T be such a tree and choose x_0 a root in T. From any $x \in T$ there exists a unique minimal path from x to x_0. Fix $n > 0$ and build a set $A_x \subset T \times \mathbb{N}$ in the following way : assign weight 1 to x (meaning put the point $x \times \{0\}$ in the set A_x) then follow the path to x_0 to the next vertex. Assign weight 1 to this vertex and keep going until either $\#A_x = n$ or you reach x_0. If x_0 is reached, assign the correct weight to x_0 so that $\#A_x = n$.

Computations show that $\#A_x \triangle A_y \leq 2d(x, y)$ and $\#A_y \cap A_y \geq n - 2d(x, y)$. Hence

$$\lim_{n\to\infty} \frac{\#A_x \triangle A_y}{\#A_y \cap A_y} = 0$$

which is enough to insure the existence of (R, ε)-A sets fo any R and ε. $\square$

2.2 Asymptotic dimension

Since proving Property A is not easy in general, we give one important criterion which insures it.

Definition 2.4 (Gromov, 2000 [Gro2])**.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of the metric space X. Given $R > 0$, the R-multiplicity of $\mathcal{U}$ is the smallest integer n such that every ball of radius R in X intersects at most n elements of $\mathcal{U}$.

The *asymptotic dimension* of X, $\mathrm{AsDim}(X)$ is the smallest integer n such that for any $R > 0$ there exists a uniformly bounded cover of X with R-multiplicity $n + 1$.

Asymptotic dimension is suited to the large scale point of view. Intuitively, we want to associate a dimension to a metric space which corresponds to the topological dimension of the space seen from afar. It shares many features with more classical notions of dimension and gives intuitive results on familiar objects (see items 1. 2. and 3. below)

Example 2.5.

1. Compact metric spaces have asymptotic dimension 0.
2. Real trees have asymptotic dimension 1.
3. $\mathrm{AsDim}(\mathbb{Z}^n) = n$.
4. Hyperbolic metric spaces have finite asymptotic dimension, but there exist hyperbolic spaces with arbitrarily large asymptotic dimension.
5. $\mathbb{Z}^{(\infty)}$ and the wreath product $\mathbb{Z} \wr \mathbb{Z}$ bith have infinite asymptotic dimension.

The following result gives a practical criterion for having property A, we state if without proof.

Theorem 2.6 (Higson-Roe, 2000 [HR])**.** *Let X be a uniformly discrete metric space. If X has finite asymptotic dimension, then X has property A.* $\square$

2.3 Coarse embeddings

Recall the following definitions :

Definition 2.7. A map $F : X \to Y$ is callled *coarse* if there exist control functions $\rho_+, \rho_- : \mathbb{R}_+ \to \mathbb{R}_+$, with $\lim_{t \to \infty} \rho_- = +\infty$, such that

$$\rho_-(d(x,y)) \leq d(F(x), F(y)) \leq \rho_+(d(x,y)), \text{ for all } x, y \in X.$$

Furthermore, the maximal map ρ_- for that condition (namely $\rho_-(t) = \inf\{d(F(x), F(y)) \mid d(x,y) \leq t\}$) is called the *compression function* of F.

The study of spaces, especially groups, which admit embeddings into Hilbert spaces (or more general Banach spaces) has been very important in connection with conjectures coming from index theory and geometry. Property A was designed by Yu as a tool to produce such embeddings.

Proposition 2.8 (Yu, 2000)**.** *Let X be a uniformly discrete metric space. If X has property A then X embeds coarsely into a Hilbert space.*

Proof: The construction is very similar to the proof of theorem 1.13. We'll define an embedding in $\bigoplus \ell^2(X \times \mathbb{N})$. First, for each $n > 0$ fix a family $\left(A_x^{(n)}\right)$ of $(n, \frac{1}{n^2})$-Følner type sets. Then define $\xi_x^{(n)} \in \ell^2(X \times \mathbb{N})$ by

$$\xi_x^{(n)} = \frac{\chi_{A_x^{(n)}}}{\sqrt{(\# A_x^{(n)})}}.$$

Now fix a base point $z \in X$ and define $F : X \to \bigoplus_n \ell^2(X \times \mathbb{N})$ by

$$F(x) = \bigoplus_{n=1}^{\infty} \left(\xi_z^{(n)} - \xi_x^{(n)}\right).$$

We need to check that this map is well-defined and is indeed a coarse embedding. Fix $x, y \in X$ and choose k minimal so that $d(x,y) \leq k+1$, we have

$$\begin{aligned}
\|F(x) - F(y)\|^2 &= \sum_{n=1}^{\infty} \left\|\xi_z^{(n)} - \xi_x^{(n)}\right\| \\
&= \sum_{n=1}^{\infty} \frac{\#\left(A_x^{(n)} \triangle A_y^{(n)}\right)}{\# A_x^{(n)}} \\
&\leq \sum_{n=1}^{k} \frac{\#\left(A_x^{(n)} \triangle A_y^{(n)}\right)}{\# A_x^{(n)}} + \sum_{n=k+1}^{\infty} \frac{1}{n^2} \\
&\leq 2k + 8 \leq 2d(x,y) + 10.
\end{aligned}$$

In the case $y = z$ this gives us that $F(x)$ is well-defined. The general statement gives an upper control function for the map F. For the lower control function, note that by condition (b) in definition 2.1, there exists a sequence S_n such that

$$\mathrm{supp}(A_x^{(n)}) \subset B(x, n)$$

It is straightforward that in order to satisfy condition (a), the sequence (S_n) must tend to infinity. Without loss of generality suppose (S_n) is increasing and define $\phi(k) = \max\{n \mid 2S_n < k \leq d(x,y)\}$, this ensures that $A_x^{(n)}$ and $A_y^{(n)}$ are disjoint whenever $n \leq \phi(k)$ We obtain

$$\begin{aligned} \|F(x) - F(y)\| &= \sum_{n=1}^{\phi(k)} \frac{\#\left(A_x^{(n)} \triangle A_y^{(n)}\right)}{\#A_x^{(n)}} + \sum_{n=\phi(k)+1}^{\infty} \frac{\#\left(A_x^{(n)} \triangle A_y^{(n)}\right)}{\#A_x^{(n)}} \\ &\geq 2\phi(k). \end{aligned}$$

□

This proposition gives us the first obstruction to property A. A space which doesn't embed coarsely into a Hilbert space can not satisfy Property A, hence families of expander graphs don't have A. Giving more examples of spaces without this property is difficult and whether the last proposition admits a converse is even harder. See A. Khukhro's notes and talk for more about the subject.

3 Quantitative properties and compression functions

The purpose of this section is to sharpen the notions of Følner and Følner type sets to obtain lower control on the compression functions of the embeddings we constructed. All following material is due to Tessera [Te1, Te2].

Definition 3.1. Let G be an amenable group, a Følner sequence $(F_n)_{n\geq 1}$ of G is called *controlled* if there exists $C > 0$ such that

$$\operatorname{diam} F_i \leq \frac{C}{\varepsilon}$$

whenever F_n is $(1,\varepsilon)$-Følner.

So, in addition to the existence of sets with small boundaries, we require that such sets can be chosen small enough. For combinatorial reasons, the condition above is the sharpest one can ask for. In other words, groups with controlled Følner sequences are as good as it gets. The following proposition shows that these groups embed in L^p spaces with very good compression functions. We provide it without proof.

Theorem 3.2 ([Te2]). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function satisfying*

$$\int_1^{\infty} \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty$$

and let G be a amenable group with controlled Følner sets. Then there exists an affine isometric action of G on a Hilbert space whose compresion function ρ satisfies

$$\rho(t) \succcurlyeq \alpha(t).$$

□

Example 3.3. Groups with polynomial growth have controlled Følner sequences. Indeed if $\#B(n) \approx n^\alpha$ it is easily checked that $\frac{\#S(n+1)}{\#B(n)} \approx 1/n$. So the family of all balls form a controlled Følner sequence.

Proposition 3.4. *The following classes of groups have controlled Følner sequences:*

1. *Polycyclic groups.*
2. *Amenable connected Lie groups.*
3. *Some algebraic semi-direct products, in particular amenable Baumslag-Solitar groups.*
4. *Wreath products of the form $F \wr \mathbb{Z}$ with F finite.*

□

The same idea applied to Følner type sets gives the following definition:

Definition 3.5. Let X be a uniformly discrete metric space, let $J : \mathbb{R}_+ \to \mathbb{R}_+$ be some increasing function and fix $1 \leq p < \infty$. We say that X has *quantitative property A(J,p)* if for each $n > 0$ there exists a family $\left(A_x^{(n)}\right)_{x\in X}$ such that

1. $\#A_x^{(n)} \geq J(n)^p$,
2. $\#\left(A_x^{(n)} \triangle A_y^{(n)}\right) \leq d(x,y)^p$,
3. $\operatorname{supp} A_x^{(n)} \subseteq B(x,n)$.

Theorem 3.6. *Let X be a metric space with property A(J,p) as above and let f be an increasing function satisfying*

$$\int_1^\infty \left(\frac{f(t)}{J(t)}\right)^p \frac{dt}{t} < \infty.$$

Then there exists a large scale Lipschitz coarse embedding of X into an L^p space with compression function ρ satisfying

$$\rho \succcurlyeq f.$$

Proof: Fix a base point $z \in X$ and fix families $\left(A_x^{(n)}\right)$ as in the definition. Define $F_n : X \to \ell^p(X)$ by

$$F_n(x) = \left(\frac{f(2^n)}{J(2^n)}\right)\left(\chi_{A_x^{(2^n)}} - \chi_{A_z^{(2^n)}}\right)$$

and set $F : X \to \left(\bigoplus \ell^p(X)\right)_p$, $F(x) = \bigoplus F_n(x)$ We need to prove that F is well-defined and that it satisfies the requirement of the theorem. We have

$$\begin{aligned}
\|F(x) - F(y)\|_p^p &= \sum_{n=1}^{\infty} \|F_n(x) - F_n(y)\|_p^p \\
&= \sum_{n=1}^{\infty} \left(\frac{f(2^n)}{J(2^n)}\right)^p \#\left(A_x^{(2^n)} \triangle A_y^{(2^n)}\right) \\
&\leq d(x,y)^p \int_1^{\infty} \left(\frac{f(2^u)}{J(2^u)}\right)^p \mathrm{d}u \\
&= d(x,y)^p \int_1^{\infty} \left(\frac{f(t)}{J(t)}\right)^p \frac{\mathrm{d}t}{t}.
\end{aligned}$$

This both shows that F is well-defined (set $y = z$) and that it is Lipschitz. On the other hand, fix $x, y \in X$ and choose N maximal such that $d(x,y) > 2^{(N+1)}$. This condition ensures that $A_x^{(2^N)}$ and $A_y^{(2^N)}$ are disjoint. We obtain

$$\begin{aligned}
\|F(x) - F(y)\|_p^p &\geq \|F_N(x) - F_N(y)\|_p^p \\
&= \left(\frac{f(2^N)}{J(2^N)}\right)^p \#\left(A_x^{(2^N)} \triangle A_y^{(2^N)}\right) \\
&\geq \left(\frac{f(2^N)}{J(2^N)}\right)^p 2J(2^N)^p \\
&= 2f(2^N) \geq 2f\left(d(x,y)\right)
\end{aligned}$$

which shows that $\rho_F \succcurlyeq f$. $\square$

We expose some classes of metric spaces for which this approach is fruitful. As in the equivariant case, looking at balls as potential controlled Følner type sets gives us results linking growth and compression functions.

Theorem 3.7. *1. Let X be a quasi-geodesic metric space with subexponential growth ν i.e.*

$$\#B(x,r) \leq \nu(r),\ \forall x \in X, r > 0.$$

Then X has $A(J_p,p)$ for every $1 \leq p < \infty$, where $J_p(t) \approx (t/\log \nu(t))^{1/p}$.

2. *Moreover, if we assume homogeneity on the size of balls, namely that*

$$\#B(x,n) < C\nu(n) \text{ for some } C > 0,$$

one can choose $J_p(t) \approx t/\log v(t)$ *for all* $1 \leq p < \infty$.

3. *Moreover, is* X *is a uniformly doubling metric space, i.e such that* ν *satisfies*

$$\nu(2r) \leq C'\nu(r),$$

then one can choose $J(t) \approx t$.

Theorem 3.8. *Let* X *be an homogeneous Riemannian manifold. Then* X *has property* $A(J,p)$ *for all* $p \geq 1$ *and* $J \approx t$.

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