PROJECT ON A SOFT INTRODUCTION TO MEASURE THEORY

In this project you will study collections of subsets (of an ambient set) that satisfy certain stability properties with respect to some elementary set-theoretic operations that you have learned and studied in class. Such a collection is called a σ-*algebra* and is the fundamental structure of an abstract mathematical theory called measure theory. Measure theory has found many applications. In particular, it provides the foundational mathematical framework for modern probability theory. Your knowledge of set theory learned in MATH 300 is sufficient to be able to study the basic properties of these σ-algebras and functions between them.

1. Measure spaces

Definition 1 (σ-algebra)**.** *Let X be a set. A collection $\mathscr{M}$ of subsets of X is called a* σ-algebra *if the following properties are satisfied:*

(Σ_1) $X \in \mathscr{M}$.

(Σ_2) *For all $A \in \mathscr{M}$ we have $X \setminus A \in \mathscr{M}$ (stability under complementation).*

(Σ_3) *For all countable collection $\{A_n\}_{n\in\mathbb{N}}$ of elements in $\mathscr{M}$ (i.e., $A_n \in \mathscr{M}$ for all $n \in \mathbb{N}$) we have $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$ (stability under countable unions).*

A set X equipped with a σ-algebra $\mathscr{M}$ is called a *measure space* and the sets in $\mathscr{M}$ are called *measurable sets*. In Exercise 1 and 2 we describe some simple examples of σ-algebras.

Exercise 1. *(6 points) Let X be a set.*

(1) *(3 points) Consider $\mathscr{M}_{trivial} = \{\emptyset, X\}$. Prove that $\mathscr{M}_{trivial}$ is a σ-algebra on X.*

(2) *(3 points) Consider $\mathscr{M}_{discrete} = \mathscr{P}(X)$. Is $\mathscr{M}_{discrete}$ a σ-algebra on X? Justify briefly your answer.*

Elements of proof. (1) Property (Σ_1) is clearly satisfied.

As for property (Σ_2), it is satisfied by looking at the complements of the two elements of $\mathscr{M}_{trivial}$.

Property (Σ_3) can be taken care off by distinguishing cases.

(2) Property (Σ_1) is also clearly satisfied here by definition of $\mathscr{P}(X)$.

Property (Σ_2) holds since the complement of a subset of X is a subset of X (definition of complement and subset).

Property (Σ_3) also holds since every countable union of subsets of X is also a subset of X (definition of arbitrary union with the index set $I = \mathbb{N}$ and subset).

□

Exercise 2. *(6 points) Let X be a set and A be a non-empty proper subset of X (i.e., $\emptyset \subset A \subset X$). Show that $\mathscr{M} = \{\emptyset, A, X \setminus A, X\}$ is a σ-algebra on X.*

Elements of proof. Property (Σ_1) is clearly satisfied.

Property (Σ_2) holds since the complements of $\emptyset$, A, $X \setminus A$, and X are X, $X \setminus A$, A, and $\emptyset$, respectively.

Property (Σ_3) holds by distinguishing cases. □

In the next exercise you are asked to prove that a σ-algebra is also stable under finite unions. This result follows by combining in a clever way property (Σ_1) with (Σ_3).

Exercise 3. *(8 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under finite unions.*

Hint: Formally, you must show that for all $k \geqslant 1$, and for all finite collection $\{A_n\}_{n=1}^k$ of elements in $\mathscr{M}$ (*i.e.,* $A_n \in \mathscr{M}$ for all $1 \leqslant n \leqslant k$) we have $\bigcup_{n=1}^k A_n \in \mathscr{M}$.

Elements of proof. Property (Σ_2) is about infinite unions and the key point is to observe that we can write a finite union as an infinite union. Let $k \geqslant 1$ and $\{A_n\}_{n=1}^k$ a finite collection of elements in $\mathscr{M}$ (*i.e.,* $A_n \in \mathscr{M}$ for all $1 \leqslant n \leqslant k$). You need to "extend" the finite sequence $\{A_n\}_{n=1}^k$ into an infinite sequence $\{\tilde{A}_n\}_{n=1}^\infty$ in a way that you do not change the finite union and that all the sets are in $\mathscr{M}$. It can be done in the following way:

$$\text{Let} \quad \tilde{A}_n = \begin{cases} A_n & \text{if } 1 \leqslant n \leqslant k \\ \emptyset & \text{if } n \geqslant k+1. \end{cases}$$

Since taking the union with the empty set does not change anything it is clear that (if it is not clear make sure you convince yourself)

$$\bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{\infty} \tilde{A}_n. \tag{1}$$

The A_n's are in $\mathscr{M}$ by assumption, but what about the empty set? Since $\emptyset = X \setminus X$ and $X \in \mathscr{M}$ it follows (from property (Σ_1) and property (Σ_2)) that $\emptyset \in \mathscr{M}$. Therefore all the $\tilde{A}_n$'s are in $\mathscr{M}$, and you can now conclude that $\bigcup_{n=1}^\infty \tilde{A}_n \in \mathscr{M}$ (by property (Σ_3)), and finally equation (1) tells you that $\bigcup_{n=1}^k A_n \in \mathscr{M}$, which is what you wanted to prove. □

Using de Morgan's laws we can prove that a σ-algebra is also stable under countable intersections.

Exercise 4. *(6 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under countable intersections.*

Hint. Formally, you must show that for all countable collection $\{A_n\}_{n=1}^\infty$ of elements in $\mathscr{M}$ (*i.e.,* $A_n \in \mathscr{M}$ for all $n \in \mathbb{N}$) we have $\bigcap_{n=1}^\infty A_n \in \mathscr{M}$. □

Elements of proof. Let $\{A_n\}_{n=1}^\infty$ be a sequence of elements in $\mathscr{M}$ (*i.e.,* $A_n \in \mathscr{M}$ for all $n \in \mathbb{N}$). You need to show that $\bigcap_{n=1}^\infty A_n \in \mathscr{M}$ and you need to rewrite this intersection in terms of the operations of complementation and union so that you can use the properties of a σ-algebra. This can be done with the de Morgan's laws. The key observation is that the following set equality holds (prove it using de Morgan's laws):

$$\bigcap_{n=1}^{\infty} A_n = \overline{\bigcup_{n=1}^{\infty} \overline{A_n}}. \tag{2}$$

Remembering that if $A_n \in \mathscr{M}$ then $\overline{A_n} \in \mathscr{M}$ (by property (Σ_1)), you can then conclude that $\bigcup_{n=1}^\infty \overline{A_n} \in \mathscr{M}$ (by invoking property (Σ_2)). Another use of property (Σ_1) tells you that $\overline{\bigcup_{n=1}^\infty \overline{A_n}} \in \mathscr{M}$, and in turn it follows from equation (2) that $\bigcap_{n=1}^\infty A_n \in \mathscr{M}$. □

As in the case of unions, stability by countable intersections implies stability by finite intersections.

Exercise 5. *(6 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under finite intersections.*

Elements of proof. Exercice 4 says that $\mathscr{M}$ is stable under infinite intersections. The deduction that the stability under finite intersection follows from the stability under infinite intersections can be carried in a very similar way as you did for unions. You will need to "extend" the finite sequence $\{A_n\}_{n=1}^k$ into an infinite sequence $\{\tilde{A}_n\}_{n=1}^\infty$ in a way that you do not change the finite intersection and that all the sets are in $\mathscr{M}$. This can be done once you observe that intersecting with the whole set X does not change anything. □

Exercise 6. *(4 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under set differences.*

Hint. Formally, you must show that for all $A,B \in \mathscr{M}$ we have $A \setminus B \in \mathscr{M}$. You could rewrite $A \setminus B$ in terms of intersection and complementation. □

Elements of proof. Let $A,B \in \mathscr{M}$. You need to observe (and prove it) that

$$A \setminus B = A \cap \overline{B}. \tag{3}$$

Equation (3) is the key point that connects the operation of set difference to the operations of complementation and intersection for which we can use the properties of the σ-algebra. Indeed, remembering that $B \in \mathscr{M}$ it follows from property (Σ_2) that $\overline{B} \in \mathscr{M}$, and then $A \cap \overline{B} \in \mathscr{M}$ by what you proved for finite intersections (since $A \in \mathscr{M}$ and $\overline{B} \in \mathscr{M}$). Therefore, $A \setminus B \in \mathscr{M}$. □

Exercise 7. *(4 points) Let X be a set. The symmetric difference between two subsets A and B of X, is defined as the set*

$$A \triangle B = \{x \in X \colon [(x \in A) \wedge (x \notin B)] \vee [(x \in B) \wedge (x \notin A)]\}.$$

If $\mathscr{M}$ is a σ-algebra on X, show that $\mathscr{M}$ is stable under symmetric differences.

Hint: Formally, you must show that if $A,B \in \mathscr{M}$ then $A \triangle B \in \mathscr{M}$. You could rewrite $A \triangle B$ in terms of intersection, union, and complementation.

Elements of proof. The proof goes more or less along the same lines as in the previous exercise once you make the crucial observation (prove it!) that

$$A \triangle B = (A \setminus B) \cup (B \setminus A). \tag{4}$$

You can conclude using what you proved for finite unions and set differences.

Alternatively, you can use the following set equality:

$$A \triangle B = (A \cap \overline{B}) \cup (B \cap \overline{A}). \tag{5}$$

In this case you can conclude using what you proved for finite unions, finite intersections, and property (Σ_2). Either way will work so pick your choice.

□

2. Creating new σ-algebras out of old ones: the pull-back procedure

Let $f \colon X \to Y$ be a function and let $\mathscr{C} \subseteq \mathscr{P}(Y)$. The collection of sets $\{f^{-1}(A) \colon A \in \mathscr{C}\}$ is a collection of subsets of $\mathscr{P}(X)$ which we will simply denote by $f^{-1}(\mathscr{C})$. The next exercise shows that if Y is equipped with a σ-algebra then there is a natural way to create a σ-algebra on X using the function f via the inverse image (this σ-algebra is called the pull-back algebra). This pull-back procedure crucially uses the Hausdorff formulas for inverse images.

Exercise 8. *(5 points) Let $f \colon X \to Y$ be a function and let $\mathscr{M} \subseteq \mathscr{P}(Y)$ be a σ-algebra on Y. Show that $f^{-1}(\mathscr{M}) = \{f^{-1}(A) \colon A \in \mathscr{M}\}$ is a σ-algebra on X.*

Elements of proof. Your goal is to show that properties (Σ_1), (Σ_2), and (Σ_3) are satisfied for the collection $f^{-1}(\mathscr{M}) = \{f^{-1}(A) \colon A \in \mathscr{M}\}$. The verification of (Σ_1) is the easiest. You need to show that $X = f^{-1}(A)$ for some $A \in \mathscr{M}$. Since it is always true that $f^{-1}(Y) = X$ (look back at the definition of the inverse image if you are in doubt), it remains to make sure that $Y \in \mathscr{M}$. But this is true by property (Σ_1) since $\mathscr{M}$ is a σ-algebra on Y.

To verify property (Σ_2) for $f^{-1}(\mathscr{M})$ you need to show that if $B \in f^{-1}(\mathscr{M})$ then $X \setminus B \in \mathscr{M}$. Note that $X = f^{-1}(Y)$, and according to the definition of $f^{-1}(\mathscr{M})$, $B = f^{-1}(D)$ for some $D \in \mathscr{M}$. So your problem reduces to showing that $X \setminus B = f^{-1}(Y) \setminus f^{-1}(D)$ is in $\mathscr{M}$. The key point is to show that following set equality holds:

$$f^{-1}(Y) \setminus f^{-1}(D) = f^{-1}(Y \setminus D). \tag{6}$$

Assuming that you have proved equality (6) (you will need to do it) you can now conclude. Indeed, $Y \setminus D \in \mathscr{M}$ by property (Σ_2) that applies to elements in $\mathscr{M}$, and thus $f^{-1}(Y \setminus D) \in f^{-1}(\mathscr{M})$ by definition of $f^{-1}(\mathscr{M})$. Therefore $X \setminus B = f^{-1}(Y) \setminus f^{-1}(D) = f^{-1}(Y \setminus D) \in f^{-1}(\mathscr{M})$, which gives the desired conclusion.

The verification of property (Σ_3) is not much different besides that you need to use a different set equality. Let $(A_n)_{n=1}^{\infty}$ be a sequence in $f^{-1}(\mathscr{M})$. You need to show that $\cup_{n=1}^{\infty} A_n$ belongs to $f^{-1}(\mathscr{M})$, *i.e.,* that $\cup_{n=1}^{\infty} A_n = f^{-1}(C)$ for some $C \in \mathscr{M}$. Since for all $n \geqslant 1$, $A_n \in f^{-1}(\mathscr{M})$ you can argue that $A_n = f^{-1}(B_n)$ for some $B_n \in \mathscr{M}$ (by definition of $f^{-1}(\mathscr{M})$). The problem boils down to showing that $\bigcup_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}(C)$ for some $C \in \mathscr{M}$. That is when Hausdorff formulas for inverse images will come to your rescue. If you can show that

$$\bigcup_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}\Big(\bigcup_{n=1}^{\infty} B_n\Big), \tag{7}$$

then since $\bigcup_{n=1}^{\infty} B_n$ is in $\mathscr{M}$ (by property (Σ_3) and since all the B_n's are in $\mathscr{M}$) the proof will be complete. It remains to prove (7), but this is elementary at this point int the semester. $\square$