# PROJECT ON A SOFT INTRODUCTION TO TOPOLOGY

In this project you will study collections of subsets (of an ambient set) that satisfy certain stability properties with respect to some elementary set-theoretic operations that you have learned and studied in class. Such a collection is called a *topology* and is the fundamental structure of an abstract mathematical theory called *abstract topology* (or point-set topology, or simply topology). Abstract topology is the mathematical framework that allows to discuss rigorously the shape of objects and the deformations that do not tear them apart, but can bend and/or stretch them. Your knowledge of set theory learned in MATH 300 is sufficient to be able to study the basic properties of these topological structures and functions between them.

## 1. Topology on a set

**Definition 1** (Topological space)**.** *Let $X$ be a set. A collection $\mathcal{O}$ of subsets of $X$ is called a* topology *on the set $X$ if the following properties are satisfied:*

($\tau_1$) *$\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$.*
($\tau_2$) *For all $A, B \in \mathcal{O}$, we have $A \cap B \in \mathcal{O}$ (stability under intersection).*
($\tau_3$) *For all index sets $I$, and for all collections $\{U_i\}_{i\in I}$ of elements of $\mathcal{O}$ (i.e., $U_i \in \mathcal{O}$ for all $i \in I$), we have $\bigcup_{i\in I} U_i \in \mathcal{O}$ (stability under arbitrary unions).*

*A set $X$ equipped with a topology $\mathcal{O}$ is called a* topological space *and the sets in $\mathcal{O}$ are called* open sets.

**Exercise 1.** *(6 points) Let $X$ be a set.*

*(1) (3 point) Consider $\mathcal{O}_{trivial} \stackrel{\text{def}}{=} \{\emptyset, X\}$. Prove that $\mathcal{O}_{trivial}$ is a topology on $X$.*
*(2) (3 point) Consider $\mathcal{O}_{discrete} \stackrel{\text{def}}{=} \mathcal{P}(X)$. Is $\mathcal{O}_{discrete}$ is a topology on $X$? Justify briefly your answer.*

Hint: You have to verify whether the collections $\mathcal{O}_{trivial}$ and $\mathcal{O}_{discrete}$ satisfy the three properties in Definition 1.

In the next exercise we show that the intersection of two topologies is a topology.

**Exercise 2.** *(6 points) Let $X$ be a set. Let $\mathcal{O}_1$ be a topology on $X$ and $\mathcal{O}_2$ be another topology on $X$ . Consider the collection of subsets of $X$, denoted $\mathcal{O}_1 \cap \mathcal{O}_2$, and defined as*

$$\mathcal{O}_1 \cap \mathcal{O}_2 \stackrel{\text{def}}{=} \{A \subseteq X : A \in \mathcal{O}_1 \text{ and } A \in \mathcal{O}_2\}.$$

*Show that the collection $\mathcal{O}_1 \cap \mathcal{O}_2$ is a topology on $X$.*

Property ($\tau_2$) about the stability under intersection of *two* sets can be extended by induction to finitely many intersections.

**Exercise 3.** *(5 points) Let $X$ be a set and $\mathcal{O}$ be a topology on $X$. Show that $\mathcal{O}$ is stable under finite intersections.*

Hint: Formally, you must show that for all $n \geqslant 1$ and for every finite collection $\{U_i\}_{i=1}^n$ of elements in $\mathcal{O}$ (*i.e.,* $U_i \in \mathcal{O}$ for all $1 \leqslant i \leqslant n$), we have $\bigcap_{i=1}^n U_i \in \mathcal{O}$; prove this statement by induction on $n$.

With the help of some remarkable set-theoretic identities, we can show that restricting a topology to a subset generates a topology on the subset.

**Exercise 4.** *(8 points) Let $X$ be a set.*

*(1) (1 point) Let $A, B, Y \subseteq X$. Show that*

$$(A \cup B) \cap Y = (A \cap Y) \cup (B \cap Y).$$

*(2) (4 points) Let $I$ be an index set, $(A_i)_{i\in I}$ be collection of subsets of $X$, and $Y \subseteq X$. Show that*

$$\Big(\bigcup_{i\in I} A_i\Big) \cap Y = \bigcup_{i\in I} (A_i \cap Y).$$

(3) *(3 points) Let $\mathcal{O}$ be a topology on X and $Y \subseteq X$. Consider the collection of subsets of X, denoted $\mathcal{O}_Y$, that is defined as*

$$\mathcal{O}_Y \stackrel{\text{def}}{=} \{A \subseteq X\colon A = B \cap Y \text{ for some } B \in \mathcal{O}\}.$$

*Show that the collection $\mathcal{O}_Y$ is a topology on Y.*

Hint: For (3) you might want to use (2) at some point in your proof.

## 2. Continuity in the abstract topological context

The classical notion of continuity that you know for real-variable real-valued functions (*i.e.,* functions from $\mathbb{R}$ to $\mathbb{R}$) can be extended to the more abstract context of topological spaces using inverse images.

**Definition 2** (Topological continuity)**.** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A function $f\colon X \to Y$ is said to be* topologically continuous *from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ if the inverse image of every open set of Y is an open set of X.*

*Formally,*
*f is topologically continuous from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ if and only if $\forall U \in \mathcal{O}_Y,\ f^{-1}(U) \in \mathcal{O}_X$.*

The goal of the next exercise is to show that topological continuity is preserved under composition.

**Exercise 5.** *(5 points) Let $X, Y, Z$ be sets equipped respectively with topologies $\mathcal{O}_X, \mathcal{O}_Y, \mathcal{O}_Z$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Show that if f is topologically continuous from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ and if g is topologically continuous from $(Y, \mathcal{O}_Y)$ to $(Z, \mathcal{O}_Z)$ then $g \circ f$ is topologically continuous from $(X, \mathcal{O}_X)$ to $(Z, \mathcal{O}_Z)$.*