PROJECT ON A SOFT INTRODUCTION TO TOPOLOGY

In this project you will study collections of subsets (of an ambient set) that satisfy certain stability properties with respect to some elementary set-theoretic operations that you have learned and studied in class. Such a collection is called a *topology* and is the fundamental structure of an abstract mathematical theory called *abstract topology* (or point-set topology, or simply topology). Abstract topology is the mathematical framework that allows to discuss rigorously the shape of objects and the deformations that do not tear them apart, but can bend and/or stretch them. Your knowledge of set theory learned in MATH 300 is sufficient to be able to study the basic properties of these topological structures and functions between them.

1. Topology on a set

Definition 1 (Topological space)**.** *Let X be a set. A collection $\mathcal{O}$ of subsets of X is called a* topology *on the set X if the following properties are satisfied:*

- *(τ_1) $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$.*
- *(τ_2) For all $A, B \in \mathcal{O}$, we have $A \cap B \in \mathcal{O}$ (stability under intersection).*
- *(τ_3) For all index sets I, and for all collections $\{U_i\}_{i\in I}$ of elements of $\mathcal{O}$ (i.e., $U_i \in \mathcal{O}$ for all $i \in I$), we have $\bigcup_{i\in I} U_i \in \mathcal{O}$ (stability under arbitrary unions).*

A set X equipped with a topology $\mathcal{O}$ is called a topological space *and the sets in $\mathcal{O}$ are called* open sets.

Exercise 1. *(6 points) Let X be a set.*

(1) (3 point) Consider $\mathcal{O}_{trivial} \overset{\text{def}}{=} \{\emptyset, X\}$. Prove that $\mathcal{O}_{trivial}$ is a topology on X.
(2) (3 point) Consider $\mathcal{O}_{discrete} \overset{\text{def}}{=} \mathcal{P}(X)$. Is $\mathcal{O}_{discrete}$ is a topology on X? Justify briefly your answer.

Hint: You have to verify whether the collections $\mathcal{O}_{trivial}$ and $\mathcal{O}_{discrete}$ satisfy the three properties in Definition 1.

Elements of proof. (1) Property (τ_1) is clearly satisfied.
As for property (τ_2), if $A, B \in \mathcal{O}_{trivial}$ you can show by distinguishing various cases that $A \cap B$ is either $\emptyset$ or X.
Another case distinction takes care of property (τ_3).
(2) Property (τ_1) is also clearly satisfied here by definition of $\mathcal{P}(X)$.
Property (τ_2) holds since every intersection of two subsets of X is a subset of X (definition of intersection and subset).
Property (τ_3) also holds since every arbitrary union of subsets of X is also a subset of X (definition of arbitrary union and subset).

□

In the next exercise we show that the intersection of two topologies is a topology.

Exercise 2. *(6 points) Let X be a set. Let $\mathcal{O}_1$ be a topology on X and $\mathcal{O}_2$ be another topology on X . Consider the collection of subsets of X, denoted $\mathcal{O}_1 \cap \mathcal{O}_2$, and defined as*

$$\mathcal{O}_1 \cap \mathcal{O}_2 \overset{\text{def}}{=} \{A \subseteq X : A \in \mathcal{O}_1 \text{ and } A \in \mathcal{O}_2\}.$$

Show that the collection $\mathcal{O}_1 \cap \mathcal{O}_2$ is a topology on X.

Elements of proof. Since $X \in \mathcal{O}_1$ and $X \in \mathcal{O}_2$ (property τ_1 of topologies) then clearly $X \in \mathcal{O}_1 \cap \mathcal{O}_2$. The same argument works for $\emptyset$.

Now if $A, B \in \mathcal{O}_1 \cap \mathcal{O}_2$ our goal is to show that $A \cap B \in \mathcal{O}_1 \cap \mathcal{O}_2$. We first show that $A \cap B \in \mathcal{O}_1$. If $A, B \in \mathcal{O}_1 \cap \mathcal{O}_2$ then $A, B \in \mathcal{O}_1$ and by property (τ_2), $A \cap B \in \mathcal{O}_1$. Arguing similarly for $\mathcal{O}_2$ we get that $A \cap B \in \mathcal{O}_2$, and thus $A \cap B \in \mathcal{O}_1 \cap \mathcal{O}_2$ (by definition of the intersection).

To verify property (τ_3) is not much different besides you start with an arbitrary collection $\{A_i\}_{i\in I}$ such that for all $i \in I$, $A_i \in \mathcal{O}_1 \cap \mathcal{O}_2$, and you need to show that $\cup_{i\in I} A_i \in \mathcal{O}_1$ and $\cup_{i\in I} A_i \in \mathcal{O}_2$. This can be done using property (τ_3) that is satisfied for $\mathcal{O}_1$ and $\mathcal{O}_2$. □

Property (τ_2) about the stability under intersection of *two* sets can be extended by induction to finitely many intersections.

Exercise 3. *(5 points) Let X be a set and O be a topology on X. Show that O is stable under finite intersections.*

Hint: Formally, you must show that for all $n \geqslant 1$ and for every finite collection $\{U_i\}_{i=1}^n$ of elements in O (*i.e.,* $U_i \in O$ for all $1 \leqslant i \leqslant n$), we have $\bigcap_{i=1}^n U_i \in O$; prove this statement by induction on n.

Elements of proof. For the base case $n = 1$ you need to show that for any set $A_1 \in O$ you have $\bigcap_{i=1}^1 A_i \in O$. This is definitely true since $\bigcap_{i=1}^1 A_i = A_1$ which is in O. For the induction step, let $n \geqslant 1$ and assume that for any sets $A_1, A_2, \ldots, A_n$ in O you have $\bigcap_{i=1}^n A_i \in O$. Then if you are given $n+1$ elements in O, say $B_1, B_2, \ldots, B_n, B_{n+1}$, you have (by associativity of intersection)

$$\bigcap_{i=1}^{n+1} B_i = \Big(\bigcap_{i=1}^{n} B_i\Big) \bigcap B_{n+1}.$$

Now, $\bigcap_{i=1}^n B_i$ is in O by the induction hypothesis (that applies to intersections with n sets), and $B_{n+1} \in O$ as well by assumption. Therefore by property (τ_3), you can conclude that $\bigcap_{i=1}^{n+1} B_i$ is in O since it can be written as the intersection of two sets in O. The conclusion follows by the principle of mathematical induction. □

With the help of some remarkable set-theoretic identities, we can show that restricting a topology to a subset generates a topology on the subset.

Exercise 4. *(8 points) Let X be a set.*

(1) (1 point) Let $A, B, Y \subseteq X$. Show that

$$(A \cup B) \cap Y = (A \cap Y) \cup (B \cap Y).$$

(2) (4 points) Let I be an index set, $(A_i)_{i \in I}$ be collection of subsets of X, and $Y \subseteq X$. Show that

$$\Big(\bigcup_{i \in I} A_i\Big) \cap Y = \bigcup_{i \in I} (A_i \cap Y).$$

(3) (3 points) Let O be a topology on X and $Y \subseteq X$. Consider the collection of subsets of X, denoted O_Y, that is defined as

$$O_Y \stackrel{\text{def}}{=} \{A \subseteq X \colon A = B \cap Y \text{ for some } B \in O\}.$$

Show that the collection O_Y is a topology on Y.

Hint: For (3) you might want to use (2) at some point in your proof.

Elements of proof. (1) Once you observe that by definition

$$x \in (A \cup B) \cap Y \iff ((x \in A) \vee (x \in B)) \wedge (x \in Y)$$

you can then use the properties of the logical connectives to deduce (after a couple of intermediate steps) that this is equivalent to $x \in (A \cap Y) \cup (B \cap Y)$.

(2) You can write a double-inclusion proof or proceed as in the question above.

(3) You must verify that the three defining properties of a topology hold. You first task is to make sure that Y can be written as an intersection of itself with an element in O. This is indeed true since Y being a subset of X, we have $Y = Y \cap X$ and $X \in O$ by definition of O. A similar argument shows that $\emptyset \in O_Y$.

The stability by intersection goes as follows. Let A_1, A_2 in O_Y. Then by definition, $A_1 = Y \cap B_1$ and $A_2 = Y \cap B_2$ for some $B_1, B_2 \in O$. By drawing a Venn diagram, you might convince yourself that (but this is not a proof and you need to provide one!)

$$A_1 \cap A_2 = (Y \cap B_1) \cap (Y \cap B_2) = Y \cap (B_1 \cap B_2). \tag{1}$$

Equation (1) is of great importance here since it will allow you to conclude. Indeed, $B_1 \cap B_2$ is in O by property (τ_2) for O and thus equation (1) tells you that $A_1 \cap A_2$ can be written as the intersection of Y with a set in O. Therefore, $A_1 \cap A_2$ is in O_Y (by definition of O_Y). To complete the proof it remains to prove the set equality (1) which you can easily do by writing a double inclusion proof or using the basic properties of the intersection.

For the last property, you can use (2). Let $\{A_i\}_{i\in I}$ such that for all $i \in I$, $A_i \in \mathcal{O}_Y$. Then by definition of $\mathcal{O}_Y$ you have $A_i = Y \cap B_i$ for some $B_i \in \mathcal{O}$. The set of interest here is $\bigcup_{i\in I} A_i = \bigcup_{i\in I}\left(Y \cap B_i\right)$, which according to (2) is equal to $Y \cap \left(\bigcup_{i\in I} B_i\right)$. Since $\bigcup_{i\in I} B_i$ is in $\mathcal{O}$ by property (τ_3) which is valid for $\mathcal{O}$, you have just showed that $\bigcup_{i\in I} A_i$ can be written as the intersection of Y with a set in $\mathcal{O}$, and the proof is complete.

□

2. Continuity in the abstract topological context

The classical notion of continuity that you know for real-variable real-valued functions (*i.e.,* functions from $\mathbb{R}$ to $\mathbb{R}$) can be extended to the more abstract context of topological spaces using inverse images.

Definition 2 (Topological continuity)**.** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A function $f : X \to Y$ is said to be* topologically continuous *from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ if the inverse image of every open set of Y is an open set of X.*

Formally,
f is topologically continuous from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ if and only if $\forall U \in \mathcal{O}_Y,\ f^{-1}(U) \in \mathcal{O}_X$.

The goal of the next exercise is to show that topological continuity is preserved under composition.

Exercise 5. *(5 points) Let X, Y, Z be sets equipped respectively with topologies $\mathcal{O}_X, \mathcal{O}_Y, \mathcal{O}_Z$. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f is topologically continuous from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ and if g is topologically continuous from $(Y, \mathcal{O}_Y)$ to $(Z, \mathcal{O}_Z)$ then $g \circ f$ is topologically continuous from $(X, \mathcal{O}_X)$ to $(Z, \mathcal{O}_Z)$.*

Elements of proof. According to Definition 2 the goal is to show that for all $A \in \mathcal{O}_Z$, $(g \circ f)^{-1}(A) \in \mathcal{O}_X$. If you were to show that the following set equality holds

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \tag{2}$$

then you would be done. Indeed $g^{-1}(A) \in \mathcal{O}_Y$ by topological continuity of g since $A \in \mathcal{O}_Z$, but then by topological continuity of f, $f^{-1}(g^{-1}(A))$ would be in $\mathcal{O}_X$ (since $g^{-1}(A) \in \mathcal{O}_Y$ as we just showed). Therefore equality $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$ will tell you that $(g \circ f)^{-1}(A) \in \mathcal{O}_X$ which was the desired conclusion. The proof of the set equality (2) is an elementary double inclusion proof that you can easily write yourself. □