REAL ANALYSIS MATH 607
HOMEWORK #10

Problem 1 (25 points). *(Various useful facts) Let $\mu, \nu \colon (X, \mathcal{M}) \to \overline{\mathbb{R}}$ be signed measures.*

1. *(3 points) If μ is a* real-measure *(i.e. for all $A \in \mathcal{M}$, $\mu(A) \in \mathbb{R}$) show that μ is bounded.*
2. *(2 points) Show that if $\mu \perp \nu$ and $\nu << |\mu|$ then $\nu \equiv 0$.*
3. *(5 points) Show that $\mu \perp \nu \iff \mu \perp |\nu| \iff (\mu \perp \nu^+)$ and $(\mu \perp \nu^-)$.*
4. *(5 points) Show that $A \in \mathcal{M}$ is a zero-set for μ iff $|\mu|(A) = 0$.*
5. *(5 points) Show that there exists an extended $|\mu|$-integrable function $f \colon (X, \mathcal{M}) \to \mathbb{R}$ such that $\mu(A) = \int_A f d|\mu|$.*
6. *(5 points) If μ is a positive measure. Let $f, g \colon (X, \mathcal{M}) \to \mathbb{R}$ be μ-integrable functions. Show that if for all $A \in \mathcal{M}$, $\int_A f d\mu = \int_A g d\mu$ then $f = g$ μ-a.e..*

Problem 2 (20 points). *(Useful facts about signed measures) Let ν be a signed measure on $(X\mathcal{M})$.*

1. *(5 points) Show that for all $A \in \mathcal{M}$, $\nu^+(A) = \sup\{\nu(B) \colon B \in \mathcal{M}, B \subset A\}$.*
2. *(5 points) Show that for all $A \in \mathcal{M}$, $\nu^-(A) = -\inf\{\nu(B) \colon B \in \mathcal{M}, B \subset A\}$.*
3. *(5 points) Show that for all $A \in \mathcal{M}$, $|\nu|(A) = \sup\{\sum_{i=1}^k |\nu(A_i)| \colon n \geqslant 1, A = \cup_{i=1}^k A_i$ with $A_i \in \mathcal{M}, A_i \cap A_j = \emptyset\, (i \neq j)\}$*
4. *(5 points) Show that if there are positive measures μ_1 ,μ_2 on $(X, \mathcal{M})$ such that $\nu = \mu_1 - \mu_2$, then $\mu_1 \geqslant \nu^+$ and $\mu_2 \geqslant \nu^-$.*

Problem 3 (25 points). *(Useful facts about absolute continuity) Let μ be a positive measure and ν be a signed measure on $(X, \mathcal{M})$.*

1. *(15 points) Show that the following assertions are equivalent:*
   1. (a) $\nu << \mu$,
   2. (b) $\nu^+ << \mu$ and $\nu^- << \mu$,
   3. (c) $|\nu| << \mu$.
2. *(5 points) If ν is moreover finite, show that $\nu << \mu \iff \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \mu(A) < \delta \implies |\nu(A)| < \varepsilon$.*
3. *(5 points) Show that if $f \colon (X, \mathcal{M}) \to \mathbb{R}$ is μ-integrable then $\forall \varepsilon > 0\ \ \exists \delta > 0\ \ \mu(A) < \delta \implies |\int_A f d\mu| < \varepsilon$*

Problem 4 (15 points). *(Extending Radon-Nikodým theorem and Lebesgue decomposition theorem)*

1. *Show that if $\nu << \mu$ then there exists a measurable and extended μ-integrable function $f \colon (X, \mathcal{M}) \to \mathbb{R}$ such that $\nu(A) = \int_A f d\mu$ under the assumption that*
   - (a) *(5 points) ν and μ are σ-finite positive measure on $(X, \mathcal{M})$.*
   - (b) *(5 points) ν is a σ-finite signed measure and μ is σ-finite positive measure on $(X, \mathcal{M})$.*
2. *(5 points) Show that if ν is a σ-finite signed measure and μ is σ-finite positive measure on $(X, \mathcal{M})$, then there exist unique signed measures ν_a and ν_s such that*
   - (a) $\nu = \nu_a + \nu_s$,
   - (b) $\nu_a << \mu$ and $\nu_x \perp \mu$.

Problem 5 (15 points)**.** *(Conditional expectation) Let* $(X,\mathcal{A},\mu)$ *be a measure space such that* $\mu(X) = 1$*. Let* $f: X \to \mathbb{R}$ *be a* μ*-integrable function and* $\mathcal{B} \subset \mathcal{A}$ *a* σ*-algebra.*

(1) (5 points) Show that there exists a $\mathcal{B}$*-measurable function* $\tilde{f}: X \to \mathbb{R}$*, called the* conditional expectation *of* f*, such that*

$$\int_B f d\mu = \int_B \tilde{f} d\mu, \text{ for all } B \in \mathcal{B}. \tag{1}$$

Moreover, show that $\tilde{f}$ *is unique (up to* μ*-a.e. equality).*

(2) (10 points) Assume now that $\mathcal{B}$ *is the* σ*-algebra generated by a finite* $\mathcal{A}$*-measurable partition of* X*, i.e.* $\mathcal{B}$ *is generated by mutually disjoint sets* $A_1,\ldots,A_n \in \mathcal{A}$ *such that* $X = \bigcup_{i=1}^k A_i$*. Give a closed formula for the conditional expectation of* f *in this situation.*