REAL ANALYSIS MATH 607
HOMEWORK #11

Problem 1. *(Old Qualifier) Let f be increasing on $[0,1]$ and*

$$g(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}, \qquad \text{for } 0 < x < 1.$$

Prove that if $A = \{x \in (0,1) : g(x) > 1\}$ then

$$f(1) - f(0) \geqslant \lambda(A),$$

where λ is the Lebesgue measure.

Problem 2. *Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Using Vitali's lemma, show that*

$$\lambda(\{x \in (a,b) : D^+ f(x) \neq D^- f(x)\}) = 0,$$

where where λ is the Lebesgue measure and for $x \in (a,b)$,

$$D^+ f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \text{ and } D^- f(x) = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}.$$

Problem 3. *Let μ be a positive measure on $(X,\mathcal{M})$. A collection of functions $(f_\alpha)_{\alpha\in A}$ is called* uniformly integrable *if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_E |f_\alpha| d\mu < \varepsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$. Show that:*

(a) Any finite subset of $L_1(\mu)$ is uniformly integrable.
(b) A sequence $(f_n)_n$ which is convergent in $L_1(\mu)$ is uniformly integrable.

Problem 4. *Let $X = [0,1]$, μ_c be the counting measure on $[0,1]$. Show that:*

(a) $\lambda << \mu_c$ but there is no measurable and nonnegative function f so that $\lambda(A) = \int_A f d\mu_c$, for all $A \in \mathcal{B}([0,1])$.
(b) μ_c has no Lebesgue decomposition with respect to λ.

Problem 5. *Assume that ν is a σ-finite signed measure, and λ and μ are σ-finite positive measures on $(X,\mathcal{M})$. Assume that $\nu << \lambda$, and $\lambda << \mu$. Define for any $h \in L_1(|\nu|)$*

$$\int_X h\, d\nu = \int_X h\, d\nu^+ - \int_X h\, d\nu^-.$$

(1) Show that $h \cdot \frac{d\nu}{d\lambda} \in L_1(\lambda)$ and

$$\int_X h(x)\, d\nu(x) = \int_X h(x) \frac{d\nu}{d\lambda}(x)\, d\lambda(x).$$

(2) Show that $\nu << \mu$ and

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu}.$$