REAL ANALYSIS MATH 607 HOMEWORK 1

Problem 1. *Let $f\colon X \to Y$*

a) *For $A, B \subset Y$, prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$*

b) *For a family $\{A_i\}_{i\in I} \subset \mathcal{P}(Y)$, show that*

$$f^{-1}\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} f^{-1}(A_i) \text{ and } f^{-1}\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} f^{-1}(A_i).$$

Give examples for the following situations:

c) $f^{-1}(f(A)) \neq A$, *for some* $A \subset X$,

d) $f(f^{-1}(B)) \neq B$, *for some* $B \subset Y$,

e) $f(\cap_{i\in I} A_i) \neq \cap_{i\in I} f(A_i)$, *for some family* $\{A_i\}_{i\in I} \subset \mathcal{P}(X)$.

Zorn's Lemma (ZL): Let $(X, \leqslant)$ be a partially ordered set. If every linearly ordered subset of X has an upper bound then X has a maximal element, i.e. there exists $x_0 \in X$ such that if $x_0 \leqslant x$ then $x = x_0$.

Well Ordering Principle (WOP): Every set can be well ordered.

Hausdorff Maximal Principal (HMP): Every partially ordered set $(X, \leqslant)$ has a maximal linearly ordered subset, i.e. there exists $S \subset X$ such that $(S, \leqslant)$ is linearly ordered but for all $S' \supsetneq S$, $(S', \leqslant)$ is not linearly ordered.

Axiom of Choice (AC): For any nonempty collection $\{X_i\}_{i\in I}$ of nonempty sets, $\Pi_{i\in I} X_i$ is nonempty.

Problem 2. *Show the following implications:*

(1) *WOP implies AC*
(2) *HMP implies ZL*
(3) *ZL implies WOP*

Problem 3. *Prove that any partial order $\leqslant$ on a set X can be extended to a linear order on the set.*

Problem 4. *Assume that a set X is not finite. Show that* $\operatorname{card}(X) \geqslant \operatorname{card}(\mathbb{N})$.

Hint: Use recursion to define an injective map $f\colon \mathbb{N} \to X$.

Problem 5.

(1) *Prove Schröder-Bernstein theorem which states that if there is an injection from X into Y and an injection from Y into X then there is a bijection between X and Y.*
(2) *Show that there is no Schröder-Bernstein theorem for continuous functions on metric spaces, i.e. find two metric spaces (M_1, d_1) and (M_2, d_2), admitting continuous injective functions: $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_1$, such that there is no bijective function $h\colon M_1 \to M_2$, so that h and h^{-1} are continuous.*

Problem 6. *Find a sequence of Riemann integrable functions* (f_n)*, defined on* $[0,1]$*, so that for all* $\varepsilon > 0$ *there is an* $n_0 \in \mathbb{N}$ *so that*

$$\int_0^1 \left| f_m(x) - f_n(x) \right| dx < \varepsilon \text{ whenever } m, n \geqslant n_0,$$

but there is no Riemann integrable function f *so that*

$$\lim_{n\to\infty} \int_0^1 \left| f(x) - f_n(x) \right| dx = 0.$$